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TESIS DOCTORAL

Essays in Microeconomic Theory

Autor:

Peter Eccles

Directores:

Ángel Hernando-Veciana y Natalia Fabra

DEPARTAMENTO DE ECONOMÍA

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Essays in Microeconomic Theory

Peter Eccles

Thesis submitted to the Department of Economics of the Universidad Carlos III de Madrid for the degree of Doctor of Philosophy, Getafe, February 2016.

Declaration

I certify that the thesis I have presented for examination for the PhD degree of the Department of Economics at the Universidad Carlos III de Madrid is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

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Abstract

My thesis considers various aspects of microeconomic theory and focuses on the different types of uncertainty that players can encounter. Each chapter studies a setting with a different type of uncertainty and draws conclusions about how players are likely to behave in such a situation.

The first chapter focuses on games of incomplete information and is joint work with Nora Wegner. We provide conditions to allow modelling situations of asymmetric information in a tractable manner. In addition we show a novel relationship between certain games of asymmetric information and corresponding games of symmetric information. This framework establishes links between certain games separately studied in the literature. The class of games considered is defined by scalable preference relations and a scalable information structure. We show that this framework can be used to solve asymmetric contests and auctions with loss aversion.

In the second chapter I move to situations in which information is almost complete. In joint work with Nora Wegner, we consider the robustness of subgame perfect implementation in situations when the preferences of players are almost perfectly known. More precisely we consider a class of information perturbations where in each state of the world players know their own preferences with certainty and receive almost perfectly informative signals about the preferences of other players. We show that implementations using two-stage sequential move mechanisms are always robust under this class of restricted perturbations, while those using more stages are often not.

The third chapter deals with a case of complete information and is joint work with

Nora Wegner. We introduce the family of weighted Raiffa solutions. An individual solution is characterised by two parameters representing the bargaining weight of each player and the speed at which agreement is reached. First we provide a cooperative foundation for this family of solutions, by appealing to two of the original axioms used by Nash and a simple monotonicity axiom. Using similar axioms we give a new axiomatization for a family of weighted Kalai-Smorodinsky solutions. Secondly we provide a non-cooperative foundation for weighted Raiffa solutions, showing how they can be implemented using simple bargaining models where offers are intermittent or the identity of the proposer is persistent. This shows that weighted Raiffa solutions have cooperative foundations closely related to those of the Kalai-Smorodinksy solution, and non-cooperative foundations closely related to those of the Nash solution.

The fourth chapter is closely related to the third chapter and is joint work with Bram Driesen and Nora Wegner. It provides a non-cooperative foundation for asymmetric generalizations of the continuous Raiffa solution. Specifically, we consider a continuous-time variation of the classic Stahl-Rubinstein bargaining model, in which each player's opportunity to make proposals is produced by an independent Poisson process, and a finite deadline ends the negotiations. Under the assumption that future payoffs are not discounted, it is shown that the payoffs realized in the unique subgame perfect equilibrium of this game approach the continuous Raiffa solution as the time horizon tends to infinity. The weights reflecting the asymmetries among the players, correspond with the Poisson arrival rates of their respective proposal processes

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Chapter 1

Scalable Games: Modelling Games of Incomplete information

1.1 Introduction

It is commonly known that there are many economic situations, where each party knows something which the other parties do not know. For instance, a company may know its cost of producing a certain product but not know the cost of its competitors. Alternatively in a common value auction a bidder may know how much he thinks the object is worth but not know the estimates of other bidders. In our analysis we will refer to these situations where each party has some private information as *games of asymmetric information*.

The main contribution of this paper is to introduce an information structure that ensures situations of asymmetric information can be modelled in a tractable manner.

The key condition we consider is on the nature of players' private information - the scalable information structure. This condition ensures that a player's private information - his signal - does not provide him with information about how his signal compares to that of other players. That is to say after observing his signal a player cannot infer anything about whether the signal observed is a relatively high or relatively low compared to those of his opponents. We say that the scalable information structure exhibits *maximal rank uncertainty*, because a player's private information does not provide him with information about the rank of his payoff type.

To illustrate this information structure, consider the following examples. Abreu and Brunnermeier (2003) study the formation of asset price bubbles. Investors learn about the existence of a bubble, but they cannot infer whether they are likely to be among the first or the last to learn about the existence of a bubble. Similarly in the double auctions studied by Satterthwaite et al. (2014), bidders and the seller do not know whether their valuation is likely to be higher or lower than that of the other bidders - and the seller. Their valuation does not provide them with information about the rank of their signal. The same information structure is also used in an application to contests and auctions with loss averse players considered in this paper.

All of the above are examples where a scalable information structure has been used to model a situation of asymmetric information which is intractable under alternative modelling assumptions. The equilibria in these examples are simple and easy to find. We refer to them as *constant strategy equilibria* as they are described by a single parameter for each player. In Abreu and Brunnermeier (2003)'s model of the formation of asset price bubbles, once he learns about the existence of the bubble, each investor decides to ride the bubble for a fixed amount of time independent of the time at which he learns. In the double auction example, all bidders shade their valuation by a constant amount, whatever their valuation. In our application to auctions and contests with loss averse bidders, players bid a constant proportion of their valuation or the effort exerted is a constant proportion of their ability respectively. The framework suggested here therefore allows us to study and make revenue comparisons for such auctions, which have received considerable attention in the literature, but cannot be solved under alternative modelling assumptions.[1]

Our entire analysis is restricted to games where players' preferences satisfy a mild

[1]See Gill and Prowse (2012), Lange and Ratan (2010) and Eisenhuth and Ewers (2015) for example.

condition, which is naturally satisfied in many situations and includes all cases where preferences are homogeneous of some degree k among others.

While the focus of our analysis lies on situations of asymmetric information, the theoretical contribution of our paper is to provide a link between games of asymmetric information and games of symmetric information.

One can think of a common situation where all parties share the same information, but there is some additional information, which cannot be accessed by any of the parties. Extreme weather events are one example of such a situation, since all parties have the same information from a central weather forecast agency but still face uncertainty. In our analysis, these situations where all parties involved have the same information but nevertheless there is some uncertainty are referred to as *games of symmetric information.*

We show that under the scalable information structure and mild restrictions on players' preferences, games of asymmetric information are strategically equivalent to games of symmetric information. Although these games differ only in what is observed by the players, they are typically used to study very different situations. Our framework therefore provides a novel link between seemingly unrelated games. The relevance of this link is illustrated by demonstrating a link between a second price auction with a reserve price where in one case participants know their valuation, but do not know the reserve price and in the other case they know the reserve price but are uncertain about their valuation. Another application using our framework to solve an otherwise complex situation considers the bargaining process to dissolve a business following bankruptcy.

The remainder of the paper is structured as follows. In the remainder of this section we relate the suggested approach to the literature. In section two, we illustrate the use of the framework in two simple examples. The key property of maximal rank uncertainty is discussed in detail section three. Section four introduces the model. The scalable structure is presented in section five, while section six contains the simplicity analysis. In section seven we present present an application to auctions with loss aversion to illustrate the simplicity and tractability of our model.

Restricting attention to settings where players' preferences can be represented by utility functions, we introduce the equivalence between scalable games of asymmetric information and scalable games of symmetric information in section eight. The relevance of this link is illustrated in section nine. Section ten concludes.

1.1.1 Related Literature

In the literature, models which satisfy the scalable preference and scalable information structure considered in this paper have been used to model specific situations of uncertainty. As mentioned above, the formation of asset price bubbles studied by Abreu and Brunnermeier (2003) and the double auctions considered by Satterthwaite et al. (2014) are two such models. Other examples include the clock games considered by Brunnermeier and Morgan (2010), as well as supply function competition studied by Vives (2011). While these papers provide models for specific situations, we aim at providing a general tool to model situations of asymmetric information.

The information structure of the proposed class of games of asymmetric information, referred to as scalable games has close links with the literature on *global games* introduced by Carlsson and Van Damme (1993) and considered in Morris and Shin (2002) and Morris and Shin (2003) among others. As in global games, players face uncertainty about the state of the world θ which is drawn from a diffuse prior. Moreover each player does not observe θ but instead receives a partially informative signal s_i about the state of the world, where $s_i = \theta + z_i$ and z_i can be interpreted as a noise term. However, in global games the main objective is equilibrium selection which arises since coordination is more difficult when the state of the world is unknown. Moreover the games considered in our paper do not necessarily have dominance regions and a player's signal typically enters his payoff function directly. Above all the focus of this paper lies on the characterization of equilibria in games of asymmetric information rather than equilibrium selection in games of complete information.

The framework proposed in our paper also has close ties with the literature on *quadratic utility models* [2] In these games there is also uncertainty about the state of the world and players receive a noisy signal of the state. Quadratic utility models typically focus on the social value of information and the role of information acquisition.[3] Applications to Cournot competition are provided by Vives (1988) and Myatt and Wallace (2013).

As in our paper, each player receives a signal about the state of the world which can be interpreted as his type and may enter his payoff function directly. The payoff function in most quadratic utility models depend on the actions of others only through the aggregate. Scalable games of asymmetric information require weaker conditions on the preference structure - for example allowing for loss aversion - at the cost of making stronger distributional assumptions on the state and the signals: the information structure in a quadratic utility model is affine, satisfying the assumption that $E[\theta|s_i] = \alpha s_i + \beta$; in the related scalable game in additive form we require the shape of the distribution to be the same for all types and hence $E[\theta|s_i] = s_i + \beta$.

In a recent paper, Morris et al. (2015) propose the concept of uniform rank belief. When there are two players, the authors say that players have a uniform rank belief if each of them assigns probability $\frac{1}{2}$ to having a higher payoff type than his opponent independent of his payoff type. Meanwhile the maximal rank uncertainty property suggested in this paper, says that the probability each player assigns to being in any particular rank is independent of his type, but it need not be equal to $\frac{1}{2}$ or $\frac{1}{n}$ in the case of n players.

Finally considering a translation from one game to a strategically equivalent game, which is easier to solve, has been proposed by Baye and Hoppe (2003) in the case of rent seeking and patent races. However they consider relationships between games of complete information, while we consider translations from a game of asymmetric information to a game of symmetric information. The aim to model situations of

[2] A comprehensive treatment of these games is provided in Angeletos and Pavan (2007) for a continuum of players, while Ui and Yoshizawa (2014) consider a discrete number of players.

[3] For models with endogenous information structures see for example Colombo and Pavan (2014) Myatt and Wallace (2012) and Pavan (2014).

incomplete information in a tractable manner is also pursued by Compte and Postlewaite (2013) who consider a private value first price auction, where bidders shade their bid by a constant amount, independent of their valuation.

1.2 Illustrative Examples

We now introduce a simple example to illustrate the strategic equivalence of certain games that are closely related, but have a different information structure. Three cases are distinguished (i) a game of asymmetric information, where a player faces uncertainty about the signals of other players, (ii) the case where players have symmetric information, but nevertheless there is some uncertainty and (iii) a complete information game.

1.2.1 Single-player Example

Consider a game, where there is one buyer wanting to buy a product. His valuation for the product is given by $s \in (0, \infty)$. The reserve price for the product is given by $\theta \in (0, \infty)$. The buyer offers to pay a fraction of his valuation $a \in \{\frac{1}{3}, \frac{1}{2}\}$. Hence, the suggested price is given by $p(s) = as$. In case the price offered by the player is higher than the reserve price, he obtains a payoff of $u(a, s, \theta) = s(1 - a)$ if$sa \geq \theta$. If the proposed price is below the reserve price the buyer obtains a payoff of zero: $u(a, s, \theta) = 0$ if $\theta > as$.

Suppose that θ is determined according to an improper prior with density function $g(\theta) = \frac{1}{\theta}$ for all $\theta \in (0, \infty)$. For any level of θ, the conditional distribution of s is given as follows:

$$f(s|\theta) = \begin{cases} \frac{1}{4} & \text{if } s = 2\theta \\ \frac{3}{4} & \text{if } s = 3\theta \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

First consider the game where the buyer observes his valuation s but does not know the reserve price θ. This is the game we refer to as *scalable game of asymmetric information*. By Bayesian updating the buyer assigns a probability of $g(\theta|s) = \frac{2}{11}$

to the case $\theta = \frac{s}{2}$ and assigns the remaining probability $g(\theta|s) = \frac{9}{11}$ to the case $\theta = \frac{s}{3}$.

The buyer's expected payoff from offering $a = \frac{1}{2}$ is given by $E\Big[u(\frac{1}{2}, s, \theta)|s\Big] = s(1 - \frac{1}{2}) = \frac{s}{2}$. Meanwhile his expected payoff from choosing $a = \frac{1}{3}$ is given by $E\Big[u(\frac{1}{3}, s, \theta)|s\Big] = \frac{9}{11}s(1 - \frac{1}{3}) = \frac{6s}{11}$. Hence, the buyer prefers to offer $a = \frac{1}{3}$ independent of his valuation.

Now consider instead the case where the buyer observes the reserve price - the price displayed at a shop - but does not know his valuation. This is the game we refer to as *scalable game of symmetric information.*

His expected payoff from offering $a = \frac{1}{2}$ is given by $E\Big[u(\frac{1}{2}, s, \theta)|\theta\Big] = \frac{1}{4}2\theta(1 - \frac{1}{2}) + \frac{3}{4}3\theta(1 - \frac{1}{2}) = \frac{11\theta}{8}$. Meanwhile his expected payoff from choosing $a = \frac{1}{3}$ is given by $E\Big[u(\frac{1}{3}, s, \theta)|\theta)\Big] = \frac{3}{4}3\theta(1 - \frac{1}{3}) = \frac{3\theta}{2}$. Again the buyer prefers to offer $a = \frac{1}{3}$ independent of the reserve price.

Moreover, note that the ratio of choosing $a = \frac{1}{3}$ to choosing $a = \frac{1}{2}$ is the same in both cases and is given by $\frac{12}{11}$. As will become clear later, this structure satisfies our scalability assumptions. We will show that these two games are equivalent.

Secondly, these games were very easy to solve. The optimal decision of a player does not depend on his valuation or the reserve price respectively. In fact any games in the class of scalable games proposed in this paper can be solved by looking at the optimal action for a buyer with a valuation $s = 1$ (or a reserve price $\theta = 1$). It is not necessary to consider the optimal decision for each valuation (reserve price) separately. This also means that the game is strategically equivalent to a game of complete information which one could choose to solve instead.

1.2.2 Multi-player Example

To further illustrate this concept and show that the framework can also capture games with several players and using a different information structure, we now present a second example.

Consider a world with two competing countries labeled $\{1, 2\}$ who actively exert their influence in a certain region. At time θ a new militant group emerges, which threatens the security of one country but furthers the interests of the other.

Each country does not immediately learn of this new development, but rather finds out at some time s_i. After learning of existence of the militant group, each country must choose how long to wait until deciding upon a response. This waiting time is denoted by $a_i \geq 0$. It is assumed that decisions are immediately put into action. Since better intelligence will lead to more effective intervention, it is assumed that the payoff associated with executing an action after waiting for a time a_i is $\overline{u}_i(a_i, s_i) = a_i$. However so that the two countries do not enter into direct conflict, only the first action chosen is executed, and the second mover receives a payoff of $\underline{u}_i(a_i, s_i) = 0$. It is assumed that countries have no prior information about when the new group will emerge, and this is modelled by θ being drawn from a diffuse prior with $g(\theta) = 1$ for all $\theta \in \mathbb{R}$. Furthermore we assume that $s_i = \theta + z_i$, where each z_i is independent of θ and is distributed uniformly over the interval $[0, 1]$. Each country observes its signal, the time at which it learns s_i, but does not observe θ.

As will be clear from the formal definition that follows this game is a scalable game of asymmetric information.

In order to solve this scalable game we look for a symmetric equilibrium in constant strategies of the form $\sigma_i(s_i) = a^*$. Since constant strategies directly imply monotonicity of reactions, the maximization problem of country i can be written as follows:

$$\max_{a_i} \int_{s_i - 1}^{s_i} a_i g(\tilde{\theta}|s_i)(1 - F(s_i + a_i - a_j|\tilde{\theta}))d\tilde{\theta}$$

The key condition we consider is on the nature of players' private information - the scalable information structure. This condition ensures that a player's private information does not provide him with information about how his payoff type compares to that of other players. That is to say after observing his signal a player cannot infer anything about whether the signal observed is a relatively high or relatively low compared to those of his opponents. The scalable information structure exhibits *maximal rank uncertainty*, because a player's private information does not provide him with information about the rank of his payoff type.

$$\int_{s_i-1}^{s_i}(1-F(s_i+a_i-a_j|\tilde{\theta}))d\tilde{\theta} = \int_{s_i-1}^{s_i} f(s_i+a_i-a_j|\tilde{\theta})d\tilde{\theta}$$

Since we are looking for a symmetric equilibrium $a_i = a_j$. Moreover from the information structure, we know that $\int_{s_i-1}^{s_i}(1-F(s_i|\tilde{\theta}))d\tilde{\theta} = 0.5$ for all s_i. Independent of its signal, each country is always equally likely to have the lower or to have the higher signal. We also know that $\int_{s_i-1}^{s_i} f(s_i|\tilde{\theta})d\tilde{\theta} = 1$ and hence $\sigma_i(s_i) = 0.5$. It turns out that this strategy is an equilibrium in constant strategies.

Consider now that instead of delay both countries learn of the emergence of the new militant group immediately and hence observe the state θ. Again each country chooses how long to wait until deciding upon its response, a_i. However in this version of the game there is a delay between the decision to act and the implementation of the action itself. This delay is given by $z_i = s_i - \theta$, where again z_i is drawn from a uniform distribution over the interval $[0, 1]$ for each $i \in \{1, 2\}$. Country i is the first mover only if $a_i + s_i < a_j + s_j$ and in this case country i receives a payoff of $\overline{u}_i(a_i, s_i) = a_i$. The second mover again receives a payoff of $\underline{u}(a_i, s_i) = 0$. The maximisation problem for each country looks as follows:

$$\max_{a_i} a_i\Big(1 - \int_{\theta}^{\theta+1} F_i(\tilde{s}_i + a_i - a_j|\theta)d\tilde{s}_i\Big)$$

Taking first order conditions leads to:

$$1 - \int_{\theta}^{\theta+1} F_i(\tilde{s}_i + a_i - a_j|\theta)d\tilde{s}_i = a_i \int_{\theta}^{\theta+1} f(\tilde{s}_i + a_i - a_j|\theta)$$

In a symmetric equilibrium we know that $a_i = a_j$. Moreover by the scalable information structure, $\int_{\theta}^{\theta+1} F_i(\tilde{s}_i + a_i - a_j|\theta))d\tilde{s}_i = 0.5$ and $\int_{\theta}^{\theta+1} f(\tilde{s}_i|\theta)d\tilde{s}_i = 1$. Therefore $\sigma_i = 0.5$ for i, j is an equilibrium of this game.

1.3 Key property

The key property driving the results in these examples is what we refer to as *maximal rank uncertainty.* We now consider this property in some more detail. Suppose there is a set of players $I = \{1, 2, ..., n\}$. Further suppose that each player i receives a signal $s_i \in \mathbb{R}$.

We now define the rank r_i of player i as follows. Let the set $\overline{I}_i = \{j : s_j \geq s_i\}$, so that $j \in \overline{I}_i$ only if the signal of player j is greater or equal to the signal of player i. With this notation in mind, define $r_i = |\overline{I}_i|$. This means that r_i captures the number of players with a signal greater than or equal to s_i. Hence if $r_i = 1$ then player i has the highest signal amongst all players and if $r_i = n$ then player i has the lowest signal amongst all players. The key property of our model can be informally stated as follows:

$$P(r_i = m|s_i) = P(r_i = m|s_i') \text{ for all } s_i, s_i', m, i$$

This equation captures that fact that the probability any player i assigns to having the n-highest signal is independent of his signal. It means that a player's signal does not give him information about his relative position compared to other players. This contrasts with a model where players have independent types. For instance consider a two player model where (i) $I = \{1, 2\}$, (ii) s_i and s_j are drawn from a uniform distribution over $[0, 1]$ and (iii) s_i and s_j are drawn independently. In this case:

$$P(r_i = 1|s_i) = s_i \text{ for all } s_i$$

This equation captures the fact that a player who observes a signal s_i close to 1 is very confident that he has the highest signal and $s_i = 1$, while a player who observes a signal s_i close to 0 is very confident that he has the lower signal and $r_i = 2$.

Hence in a model with independent types players gain *rank information* about their relative position compared to other players when they observe their signal. Meanwhile the key property in our model ensures that players do not gain *rank* *information* from observing their signal. We believe this to be a more appropriate way to model certain situations such as some auctions where a bidder's valuation may not help him decide whether he has the highest valuation or not (this would be the case in situations where having a higher valuation increases the likelihood that other bidders also have a high valuation).

1.4 The Model

We now introduce the general model and formally define a class of games with maximal rank uncertainty, capturing the illustrative examples above.

Consider a finite set of players $I = \{1, ..., n\}$. The state is denoted by $\theta \in \Theta = (\underline{\theta}, \overline{\theta})$ and each player is associated with a signal $s_i \in S_i = (\underline{s}_i, \overline{s}_i)$. In most applications this signal can be thought of as a player's type and hence describing his preferences. For simplicity we consider $S_i = \Theta$ for all $i \in I$. We expect the results to hold for any open interval S_i.

Each player i simultaneously and independently chooses an action $a_i \in A_i \subseteq \mathbb{R}$. Action sets may be player specific. To ease notation we use $\mathbf{s} = (s_1, \ldots, s_n)$ to denote the vector of players' signals and $\mathbf{a} = (a_1, \ldots, a_n)$ to denote the vector of players' actions. Moreover we define $\omega = (\mathbf{a}, \theta, \mathbf{s})$ to be an outcome described by a vector of actions $\mathbf{a}$ the state θ and the vector of all players signals $\mathbf{s}$. Let Ω denote the set of outcomes.

In order to cover both expected and certain non-expected utility frameworks, we state players' preference relations in terms of lotteries over outcomes. A lottery $L \in \mathbb{L}$ is a cumulative distribution over the outcomes Ω, $L : \Omega \mapsto [0, 1]$, where an outcome ω is given by $(\mathbf{a}, \theta, \mathbf{s})$. Each player has a preference relation over the set of lotteries $\mathbb{L}$. It is assumed that these preference relations are complete, continuous and transitive, but crucially we do not assume the independence axiom.

The information structure is given as follows. The distribution of each player's signal s_i is given by $F_i(s_i^*|\theta)$. These distributions are assumed to be independent conditional on θ and $F(\mathbf{s}|\theta)$ denotes the distribution of all players' signals conditional on θ. In addition we assume that these conditional distributions have a density, which we denote by $f_i(s_i|\theta)$.

It is crucial in our analysis that we allow θ to have an improper prior distribution. To this aim, we define the prior over θ with a function $g : \Theta \to [0, \infty)$. The probability that $\theta \leq \theta^*$ given s_i, $G(\theta^*|s_i)$ is given as follows:

$$G_i(\theta^*|s_i) = \int_{\underline{\theta}}^{\theta^*} \frac{f_i(s_i|\theta)g(\theta)}{\int_\Theta f_i(s_i|\tilde{\theta})g(\tilde{\theta})d\tilde{\theta}} d\theta. \tag{1.2}$$

The case of a proper prior corresponds to the case in which (i) g plays the role of a density and (ii) (1.2) is the standard Bayes rule. However, the above formulation also allows for improper priors in which $\int_\Theta g(\theta)d\theta = \infty$. We use $g_i(\theta^*|s_i)$ to denote the probability density function corresponding to $G_i(\theta^*|s_i)$.

To simplify notation, we use Γ to summarise the primitives of the model:

$$\Gamma \equiv \{I, \Theta, (A_i)_{i\in I}, (\succeq_i)_{i\in I}, g, (F_i)_{i\in I}\}$$

Two cases of this basic model are studied in our paper. The difference lies in the source of the uncertainty. First we consider the case where each player privately observes his signal s_i, but does not observe the state. This is the game we refer to as a *game of asymmetric information*. Second we consider the case where all players observe the state, but do not observe their private signals s_i. This is the game we refer to as a *game of symmetric information*.

We write $\mathcal{A}(\Gamma)$ to denote the game of asymmetric information where each player privately observes s_i while θ is not observed. Similarly $\mathcal{S}(\Gamma)$ is used to denote the game of symmetric information where $\theta$is commonly known among all players, but the signals s_i are not observed.

1.4.1 Strategies

In order to avoid introducing additional notation, we will jointly define the strategies used in games of asymmetric information and games of symmetric information, despite the differences in what is observed by the players.

A strategy for player i is described by a cumulative distribution function over actions, conditional on the state θ and on the player's signal s_i and is denoted by $\sigma_i(a_i, \theta, s_i)$. This notation allows us to capture both mixed strategies and pure strategies succinctly. We use σ to denote $(\sigma_1, \dots, \sigma_n)$.

In a game of asymmetric information $\mathcal{A}(\Gamma)$, players do not observe the state θ and hence feasible strategies are constant in θ. The set of strategies which are constant in θ is denoted by $\Sigma^{\mathcal{A}}$, as it is the set of feasible strategies under asymmetric information: $\sigma_i^{\mathcal{A}}(a_i, \theta, s_i) = \sigma_i^{\mathcal{A}}(a_i, \theta', s_i)$ for all θ, θ', s_i and a_i. A typical element in this set for player i is denoted by $\sigma_i^{\mathcal{A}}$.

Similarly, in a game of symmetric information $\mathcal{S}(\Gamma)$, players do not observe their signal s_i, and we require the strategy to be constant in s_i. The set of such strategies is denoted by $\Sigma^{\mathcal{S}}$ and describes the feasible strategies in a game of symmetric information: $\sigma_i^{\mathcal{S}}(a_i, \theta, s_i) = \sigma_i^{\mathcal{S}}(a_i, \theta, s_i')$ for all θ, s_i, s_i' and a_i. A typical element in this set for player i is denoted by $\sigma_i^{\mathcal{S}}$.

Our analysis makes use of strategies which are constant in both s_i and θ. We refer to these strategies as *constant strategies*. The set of these strategies is given by Σ^C and a typical element in this set for player i is denoted σ_i^C: $\sigma_i^C(a_i, \theta, s_i) = \sigma_i^{\mathcal{A}}(a_i, \theta', s_i')$ for all $\theta, \theta', s_i, s_i'$ and a_i. A typical element in this set for player i is denoted by $\sigma_i^{\mathcal{A}}$.

In the examples mentioned in the introduction, these constant strategies correspond to all investors riding the bubble for a fixed amount of time, shading their valuation by a constant amount in the double auction or bidding a constant fraction of their valuation respectively.

1.4.2 Equilibria

Suppose (i) player i has observed a signal s_i^*, (ii) player i chooses an action $a_i \in A_i$ and (iii) other players play according to the strategy profile $\sigma^{\mathcal{A}}$ where $\sigma^{\mathcal{A}}(\mathbf{a}|\mathbf{s}) = \prod_{j \in I} \sigma^c(a_j)$. We define $L_i^{\mathcal{A}}[s_i^*; a_i, (\sigma_j^c)_{j \neq i}]$ to capture the weights that player i assigns to different outcomes in this situation. Hence:

$$L_i^{\mathcal{A}}[s_i^*; a_i, (\sigma_j^c)_{j\neq i}](\mathbf{a}, \mathbf{s}, \theta) = \begin{cases} \left(\prod_{i\neq j} \sigma_j^c(a_j)\right) \int_{\underline{\theta}}^{\theta} \int_{\underline{\mathbf{s}}_{-\mathbf{i}}}^{\mathbf{s}_{-i}} g(\tilde{\theta}|s_i^*) f(\tilde{\mathbf{s}}_{-i}|\tilde{\theta}) d\tilde{\mathbf{s}}_{-i} d\tilde{\theta} \text{ if } s_i \geq s_i^* \text{ and } a_i \geq a_i^* \\ 0 \text{ otherwise} \end{cases}$$

The equilibrium for a game of asymmetric information $\mathcal{A}(\Gamma)$ can now be defined as follows:

Definition 1 (Constant strategy equilibrium: Game of asymmetric information)**.** *A strategy profile $\sigma^{\mathcal{A}} \in \Sigma^{\mathcal{A}}$ is a constant strategy equilibrium of the game $\mathcal{A}(\Gamma)$ if for all players $i \in I$ and for all signals $s_i^* \in S_i$ (i) $\sigma^{\mathcal{A}}(\mathbf{a}|\mathbf{s}^*) = \prod_{j\in I} \sigma_j^c(a_j)$ and (ii) for all actions $a_i^* \in supp(\sigma_i^c)$ and all deviations $\hat{a}_i \in A_i$ it holds that:*

$$L_i^{\mathcal{A}}[s_i^*; a_i^*, (\sigma_j^c)_{j\neq i}] \succeq_i L_i^{\mathcal{A}}[s_i^*; \hat{a}_i, (\sigma_j^c)_{j\neq i}]$$

This definition says that the constant strategy profile $\sigma^{\mathcal{A}}$ is an equilibrium, if each player i - given that he observes signal s_i^* - weakly prefers the lottery generated by choosing choosing any optimal action $a_i^* \in$ suppσ_i^c compared to the lottery generated by choosing any alternative action $\hat{a}_i$. Although this definition only considers constant strategy profiles, it allows for arbitrary deviations. Hence a constant strategy equilibrium of $\mathcal{A}(\Gamma)$ is also a Bayesian Nash equilibrium of $\mathcal{A}(\Gamma)$.

Suppose now that (i) the state is known to be θ^*, (ii) player i chooses action a_i^* and (iii) other players play according to the strategy profile $\sigma^S(\mathbf{a}|\theta^*)$ where $\sigma^S(\mathbf{a}|\theta^*) = \prod_{j\in I} \sigma_j^c(a_j)$. We define $L_i^{\mathcal{S}}[\theta^*; a_i^*, (\sigma_j^c)_{j\neq i}]$ to capture the weights that player i assigns to different outcomes in this situation. Therefore:

$$L_i^{\mathcal{S}}[\theta^*; a_i, (\sigma_j^c)_{j\neq i}](\mathbf{a}, \mathbf{s}, \theta) = \begin{cases} \left(\prod_{i\neq j} \sigma_j^c(a_j)\right) \int_{\underline{\mathbf{s}}}^{\mathbf{s}} f(\tilde{\mathbf{s}}|\theta^*) d\tilde{\mathbf{s}} \text{ if } \theta \geq \theta^* \text{ and } a_i \geq a_i^* \\ 0 \text{ otherwise} \end{cases}$$

A constant strategy equilibrium in this game of symmetric information $\mathcal{S}(\Gamma)$, can now be defined as follows:

Definition 2 (Constant strategy equilibrium: Game of symmetric information)**.** *A strategy profile $\sigma^{\mathcal{S}} \in \Sigma^{\mathcal{S}}$ is a constant strategy equilibrium of the game $\mathcal{S}(\Gamma)$ if* *for all states* $\theta^* \in \Theta$ *and for all players* $i \in I$ *(i)* $\sigma^{\mathcal{S}}(\mathbf{a}|\theta^*) = \prod_{j\in I} \sigma_j^c(a_j)$ *and (ii) for all actions* $a_i^* \in supp(\sigma_i)$ *and all deviations* $\hat{a}_i \in A_i$ *it holds that:*

$$L_i^{\mathcal{S}}[\theta^*; a_i^*, (\sigma_j^c)_{j\neq i}] \succeq_i L_i^{\mathcal{S}}[\theta^*; \hat{a}_i, (\sigma_j^c)_{j\neq i}]$$

This definition says that the constant strategy profile $\sigma^{\mathcal{S}}$ is an equilibrium, if each player i - given that the state is θ - weakly prefers the lottery generated by choosing choosing any optimal action $a_i^* \in$ suppσ_i compared to the lottery generated by choosing any alternative action $\hat{a}_i$. Again note that while this definition only considers constant strategy profiles, it allows for arbitrary deviations and hence constant strategy equilibria of $\mathcal{S}(\Gamma)$ are also Bayesian Nash equilibria of $\mathcal{S}(\Gamma)$. In the next section we propose conditions on the preference relation and information structure which help ensure that constant strategy equilibria exist.

1.5 Scalable primitives

We now propose conditions on the players' preference relations and the information structure which lead to games with the desired scalability properties.

In order to define scalable games in a general framework, we use a generator and an operator to state the required conditions. A *generator*, denoted by H, is a strictly increasing bijection from Θ to $\mathbb{R}$. We also assume that it is differentiable.[4] Secondly the *operator* associated to the generator H, denoted by $\oplus_H$, maps any two numbers $(a, b) \in \Theta^2$ into the unique number $a \oplus_H b \in \Theta$ that solves:

$$a \oplus_H b \equiv H^{-1}\Big(H(a) + H(b)\Big)$$

The operator $\ominus_H$ is defined symmetrically as $a \ominus_H b \equiv H^{-1}\Big(H(a) - H(b)\Big)$. [5] An obvious example is $H(x) = x$. In this case, the operators $\oplus_H$ and $\ominus_H$ are the usual sum and subtraction, respectively. Another example is the case when

[4]The assumption of differentiability is made in order to simplify calculations. We believe that this assumption is not necessary.

[5]The terms $\oplus_H$ and $\ominus_H$ can be thought of the the normal $+$ and $-$ after a projection of the state space from $\mathbb{R}$ to Θ.

Table 1.1. The generator

Θ	$H(\theta)$	$a \oplus_H b$	$a \ominus_H b$
$\mathbb{R}$	θ	$a+b$	$a-b$
$\mathbb{R}_{++}$	$\frac{1}{\theta}$	$a \times b$	$a \div b$

$H(x) = \ln(x)$. Here the operators $\oplus_H$ and $\ominus_H$ are the usual multiplication and division, respectively.
In some cases it is useful to consider a reference point for either the signal of player i or the state. For a generator H, we use $0_H := x$ such that $H(x) = 0$.[6] Returning to the illustrative examples, the reference point in the case of a single buyer wanting to buy an object would be his valuation $s = 1$, or the reserve price $\theta = 1$, while in the two country example, the reference point corresponds to the case where a firm learns about the existence of the military group at time zero or where the military group is formed at time zero.

1.5.1 Scalable preference relations

Given an outcome $\omega = (\mathbf{a}, \theta, \mathbf{s})$,let $\omega \oplus_H k \equiv (\mathbf{a}, \theta \oplus_H k, \mathbf{s} \oplus_H k)$ and let $[L \oplus_H k](w) \equiv L(w \oplus_H k)$. This allows us to introduce scalable preference relations.

Definition 3 (Scalable preference relations)**.** *A preference relation $\succeq_i$ is scalable with respect to H if whenever:*

$$L_i \succeq_i L'_i$$

then,

$$[L_i \oplus_H k] \succeq_i [L'_i \oplus_H k]$$

This definition says that if player i prefers lottery L_i to the lottery L'_i, then he will also prefer the lottery corresponding to scaling up the signals of all players and the state by some constant k using the notion of scalability given by H and keeping all actions constant, in lottery L, to a similarly scaled version of the lottery L'. This preference structure is naturally satisfied in many situations. All

[6]The choice of this reference point is arbitrary and has no particular meaning.

examples mentioned in the introduction - including auctions and contests with loss averse players - exhibit scalable preference relations. Moreover this general preference structure allows us to capture and hence study situations which cannot be modelled using expected utility, such as the auctions with loss averse players. However this general structure is not always necessary. We will later present a simple sufficient condition for it to be satisfied.

1.5.2 Scalable information structure

The second key element of our analysis is the information structure. A scalable information structure is defined as follows:

Definition 4 (Scalable information structure)**.** *The information structure* $\{g, \{F_i(s_i|\theta)\}_{i\in I}\}$ *is scalable with respect to* H *if:*

1. $g(\theta) = H'(\theta)$ *for all* $\theta \in \Theta$
2. *For all* $\theta, k, s_i \in \Theta$
$$F_i(s_i|\theta) = F_i(s_i \oplus_H k|\theta \oplus_H k)$$

The first part of this definition ensures that the notion of scalability used in the information structure corresponds is appropriate for the primitives.

Part two of this definition captures the fact that the conditional distribution of signals has a similar shape when θ is changed. When $a \oplus_H b = a + b$ this implies that conditional beliefs are additively invariant: that is to say the shape of the distribution is common knowledge but players do not know their position in the distribution. For instance this holds when players know that the distribution is uniform over the interval $[\theta - 1, \theta + 1]$, but do not necessarily know the value of the state θ. This is illustrated in Figure 1.1.

Meanwhile when $a \oplus_H b = a \times b$ this definition implies that conditional beliefs are homogeneous of degree 0. For instance this holds when players know that the distribution is uniform over the interval $[0, 2\theta]$, but do not necessarily know the value of the median θ. This is illustrated in Figure 1.2.

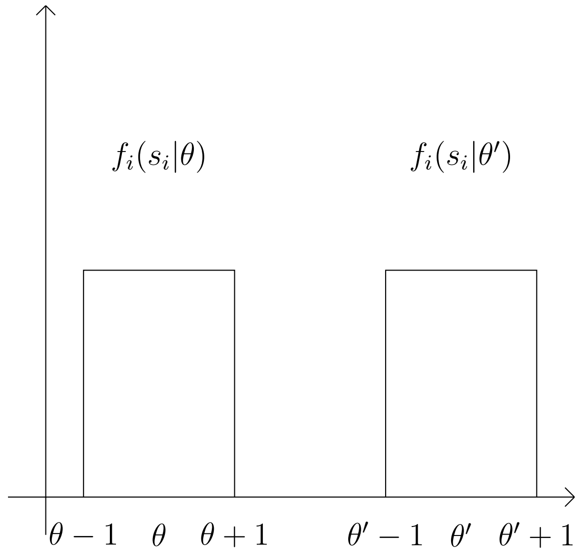


Figure 1.1. Uniform:Additive

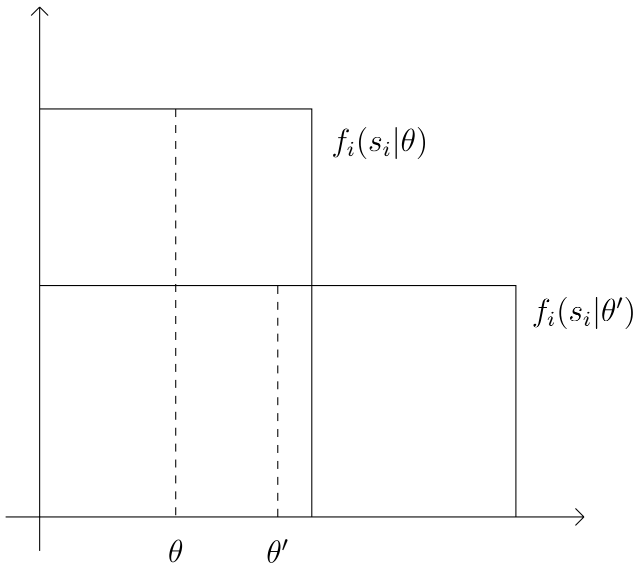


Figure 1.2. Uniform: Multiplicative

1.5.3 Scalable Games

Considering a structure with primitives given by Γ, where players simultaneously choose an action, we say that the structure is *scalable* if the preference relations are scalable (see definition 3.5.1) and the information structure is scalable (see definition 4).

More precisely, combining the notion of a scalable information structure with the definition of $\mathcal{S}(\Gamma)$ and $\mathcal{A}(\Gamma)$, we define the following:

Definition 5 (Scalable game of asymmetric information)**.** *We say that the game of asymmetric information $\mathcal{A}(\Gamma)$ is a scalable game of asymmetric information, if the preferences $(\succeq_i)_{i\in I}$ are scalable (see definition 3.5.1) and the information structure $\{g,(F_i)_{i\in I}\}$ is scalable (see definition 4).*

Definition 6 (Scalable game of symmetric information)**.** *We say that the game of symmetric information $\mathcal{S}(\Gamma)$ is a scalable game of symmetric information, if the preferences $(\succeq_i)_{i\in I}$ are scalable (see definition 3.5.1) and the information structure $\{g,(F_i)_{i\in I}\}$ is scalable (see definition 4).*

These are the two types of games to which we apply our framework.

1.6 Analysis: Simplicity

In this section we show that scalable games are particularly tractable. This is demonstrated by drawing the connection between scalable games of asymmetric information $\mathcal{A}(\Gamma)$ and an associated game of complete information.

Informally, scalable games are particularly easy to solve, because in order to determine the optimal strategy for each player, it is sufficient to look at one particular signal or state for each player. The optimal actions are the same, when he has a different signal, or the state is different. In the case of pure strategies, the problem is reduced from solving for a fixed point in the space of functions to solving for a fixed point in the space of vectors, one for each player.

This simplicity can also be demonstrated by considering a related game of complete information.

Let $L_i^{\mathcal{A}}[\sigma^c, 0_H]$ represent the lottery that player i assigns to possible outcomes when (i) player i observes signal $s_i = 0_H$ and (ii) players play according to constant strategy profile $\sigma(\mathbf{a}|\mathbf{s}) = \sigma^c(\mathbf{a})$. As a reminder:

$$L_i^{\mathcal{A}}[\sigma^c, 0_H](\mathbf{a}, s_i, s_{-i}, \theta) = \begin{cases} \sigma^c(\mathbf{a}) \int_\theta \int_{s_{-i}} g(\theta|0_H) f(s_{-i}|\theta) ds_{-i} d\theta \text{ when } s_i \geq 0_H \\ 0 \text{ otherwise} \end{cases}$$

Using this notation we can now define the complete information game $\mathcal{C}(\Gamma)$:

Definition 7 (Complete information game $\mathcal{C}(\Gamma)$)**.** *The complete information game corresponding to the primitives* Γ *is given by* $\mathcal{C}(\Gamma) := \{I, (A_i)_{i \in I}, (\succeq_i^c)_{i \in I}\}$ *where:*

$$\sigma^c \succeq_i^c \hat{\sigma}^c \text{ if and only if } L_i^{\mathcal{A}}{}_{[\sigma^c]} \succeq_i L_i^{\mathcal{A}}{}_{[\hat{\sigma}^c]}$$

This leads us to the following result:

Proposition 1.6.1 (Game of complete information)**.** *For goven primitives* Γ*, the constant strategy profile* σ^c *is a Bayesian Nash equilbrium of the scalable game* $\mathcal{A}(\Gamma)$*, if and only if it is a Nash equilibrium of the complete information game* $\mathcal{C}(\Gamma)$*.*

Hence a scalable game where players have a symmetric information - denoted by $\mathcal{A}(\Gamma)$ - is particularly easy to solve because it is sufficient to study a corresponding game of complete information $\mathcal{C}(\Gamma)$. If we also require that (i) the independence axiom holds (so that preferences can be represented by a utility function) and (ii) each action space A_i is finite then it is a standard result that the complete information game $\mathcal{C}(\Gamma)$ must have an equilibrium (possibly in mixed strategies). Then by appealing to proposition 1.6.1, we can ensure that an equilibrium in constant strategies also exists in the game of asymmetric information $\mathcal{A}(\Gamma)$. Even when these conditions do not hold, an equilibrium can often be found by studying the game of complete information $\mathcal{C}(\Gamma)$.

Being able to focus on equilibria in constant strategies, makes scalable games easier to solve than more general games of incomplete information. When looking for an equilibrium in constant strategies, the problem reduces from looking for a fixed point in the space of functions to finding a fixed point in the space of vectors.

1.7 Applications: Simplicity

To show that the simplicity of the scalable game framework allows to study situations which are difficult to model under alternative assumptions, we now present an application to auctions and contests with loss averse participants.

1.7.1 Auctions and Contests with loss averse players

We now study how loss aversion affects the bidding behaviour - or respectively the effort exerted - in auctions or contests. In particular we compare the effects of loss aversion in first price auctions and all pay auctions.

Consider a contest with I participants, where one prize is to be handed out. Player i's valuation of the prize is given by his signal s_i. Each contestant's effort is denoted by a_i and is interpreted as the proportion of his valuation he spends. The outcome function is given as follows:

$$\phi_i(\mathbf{a}, \mathbf{s}, \theta) = \begin{cases} 1 & \text{if } a_i s_i \geq a_j s_j \text{ for all } j \neq i \\ 0 & \text{otherwise} \end{cases} \tag{1.3}$$

We consider information structures $\{g, F(\mathbf{s}|\theta)\}$ which are scalable according to definition 3.5.1. In the scalable game of asymmetric information the probability player i assigns to winning the contest and hence the contest success function is denoted $\psi(a_i, a_{-i})$. Assuming that contestants choose constant efforts, this function is independent of s_i and is given as follows:

$$\psi_i(a_i, a_{-i}) = \int_{-\infty^n}^{\mathbf{s}} (\phi_i(\mathbf{a}, \tilde{\mathbf{s}}, \theta)|\theta) d\tilde{\mathbf{s}} \tag{1.4}$$

Note that this function is strictly increasing in a_i given a_{-i}. Given a vector of actions $\mathbf{a}$, a contestant is equally likely to win, independent of his signal.

The analysis extends to any contest success function $\psi_i(a_i, a_{-i})$ which is strictly increasing in a_i given a_{-i} and does not depend on s_i, but is not necessarily derived from a deterministic allocation rule.

In addition we assume that players are loss averse. In particular, players feel a loss, whenever their true payoff is lower than their expected payoff. This loss is given by β times the expected payoff minus the actual payoff, whenever this is positive:

$$u_i = \begin{cases} \pi_i - \beta(E(\pi_i) - \pi_i) & \text{if } \pi_i \leq E(\pi_i) \\ \pi_i & \text{otherwise} \end{cases} \tag{1.5}$$

A related definition of expectation based loss aversion is considered in Koszegi and Rabin (2006).

1.7.2 Loss Aversion in standard contests

First we consider a standard contest, where each contestant pays his effort. Player i's expected utility is denoted $V(a_i, a_{-i}|\beta)$ and is given as follows:

$$V(a_i, a_{-i}|\beta) = \psi_i(a_i, a_{-i})s_i - as_i - \beta\Big[1 - \psi_i(a_i, a_{-i})\Big]\Big[\psi_i(a_i, a_{-i})s_i\Big] \tag{1.6}$$

Note that the corresponding distribution over outcomes is scalable (see definition 4). Since we have assumed a scalable information structure (see definition 3.5.1), this is a scalable game of asymmetric information. We now show that this problem is indeed tractable.

First differentiating with respect to a_i observing that in equilibrium $\frac{\delta V(a_i,a_{-i}|\beta)}{\delta a_i} = 0$, we consider the effect of changes in the degree of loss aversion β:

$$\frac{\delta V}{\delta \beta \delta a_i} = 2\Big[\psi_i(a_i, a_{-i}) - \frac{1}{2}\Big] \tag{1.7}$$

Since V is a single-peaked function. Hence at equilibrium $\frac{\delta V}{\delta a_i} = 0$. If $\psi_i(a_i, a_{-i}) > \frac{1}{2}$, then an increase in β will increase $\frac{\delta V}{\delta a_i} = 0$. In order to remain in equilibrium a_i will increase. On the other hand if $\psi_i(a_i, a_{-i}) < \frac{1}{2}$, the opposite effect prevails and a_i will decrease. This leads to the following proposition:

Proposition 1.7.1. *In a contest where all players pay their effort and which is a scalable game of asymmetric information $\mathcal{A}(\Gamma)$, if players become more loss averse* *(ie β increases), then*

- *Players with over a half chance of winning will bid higher.*
- *Bidders with under a half chance of winning will bid lower.*

1.7.3 First price auction with loss aversion

Now consider the case of a contest, where players do not have to pay their cost and hence a generalised first price auction with loss averse players. The distribution over outcomes remains scalable and the problem is still tractable. In this case, the expected utility is given as follows:

$$\begin{aligned} V(a_i, a_j|\beta) &= \psi_i(a_i, a_j)\Big[s_i - as_i\Big] - \beta\Big[1 - \psi_i(a_i, a_j)\Big]\Big[\psi_i(a_i, a_j)(s - as_i)\Big] \\ &= \Big[(1 - a_i)\psi_i(a_i, a_j)\Big]\Big[1 - \beta\psi_i(a_i, a_j)\Big] \end{aligned}$$

Differentiating with respect to a_i and noting that in equilibrium $\frac{\delta V}{\delta a_i} = 0$ gives:

$$a_i = 1 - \frac{\psi_i(a_i, a_j) - \beta\psi_i(a_i, a_j)^2}{\frac{\delta\psi_i(a_i, a_j)}{\delta a_i}} \tag{1.8}$$

Moreover in a symmetric first price auction, the player with highest bid receives the object and $a_i = a_j$. Hence:

$$a_i = 1 - \frac{\frac{1}{2} - \beta\frac{1}{4}}{\delta\psi_i(a_i, a_i)\delta a_i} \tag{1.9}$$

An increase in β leads to an increase of the right hand side. Therefore for the first order condition to continue to hold, a_i must increase. This leads to the following result:

Proposition 1.7.2. *In a generalised first price auction which is a scalable game of asymmetric information, where players are loss averse, then: If players become more loss averse (ie β increases) they choose a higher effort (bid).*

Using propositions 1.7.1 and 1.7.2 we can determine the optimal strategy for a loss averse bidder participating in a general contest or a first price auction respectively. Although due to the improper prior, the seller's expected revenue cannot be computed under the scalable information structure, it can be calculated for any given state θ. Propositions 1.7.1 and 1.7.2 can therefore be used to determine a seller's - and buyers' - preferred auction mechanism when players are loss averse.

1.8 Analysis: Equivalence

In many applications players' preferences satisfy the independence axiom. This means that players have expected utility and their preferences can be represented by von Neumann-Morgenstern utility functions. We now provide a sufficient condition for preferences to satisfy definition 3.5.1 when players are expected utility maximisers. This expected utility representation will also be used to show the equivalence between games of asymmetric information and games of symmetric information.

We denote the von Neumann-Morgenstern utility function of player i by $u_i(\mathbf{a}, \theta, \mathbf{s})$, where $\mathbf{a}$ is the vector of players' actions, θ is the state and $\mathbf{s}$ is the vector of players' signals.

Attention is limited to utility functions which satisfy the following:

Assumption 1 (Scalable payoff structure)**.**

$$g(\theta)u_i(\mathbf{a}, \mathbf{s}, \theta) = g(\theta \oplus_H k)u_i(\mathbf{a}, \mathbf{s} \oplus_H k, \theta \oplus_H k) \textit{ for all } i \in I$$

It is clear that if a utility function satisfies assumption 1, then the corresponding preference relation over lotteries are scalable (see definition 3.5.1). Therefore utility functions that satisfy assumption 1 are a special case of the more general class of preferences studied in the previous section. Assumption 1 is satisfied when $H(\theta) = \theta$, the operator $\oplus_H$ represents $+$ and:

$$u_i(\mathbf{a}, \mathbf{s}, \theta) = u_i(\mathbf{a}, \mathbf{s} + k, \theta + k)$$

Moreover, assumption 1 is also satisfied when $H(\theta) = \ln(\theta)$, the operator $\oplus_H$ represents $\times$ and:

$$u_i(\mathbf{a}, \mathbf{s}, \theta) = \frac{1}{k} u_i(\mathbf{a}, \mathbf{s}.k, \theta.k)$$

Hence assumption 1 holds for utility functions which are (i) homogeneous of degree 0 in the log transform and (ii) homogeneous of degree 1. In particular, it is satisfied by all the examples using utility functions given in this paper. This includes (i) beauty contests and quadratic utility models where utility functions are homogeneous of degree 0 in the log transform and (ii) first price, second price and all pay auctions with risk neutral bidders where utility functions are homogeneous of degree 1. Using this assumption, we can now state the main result of this paper:

Theorem 1.8.1. *Suppose $\mathcal{A}(\Gamma)$ is a scalable game of asymmetric information and $\mathcal{S}(\Gamma)$ is a scalable game of symmetric information with the same primitives Γ. If preferences in $\Gamma^{\mathcal{S}}$ satisfy assumption 1, then the strategy profile σ^C is a Nash equilibrium of $\mathcal{A}(\Gamma)$ if and only if it is a Nash equilibrium of $\mathcal{S}(\Gamma)$.*

The proof can be found in the appendix.

This result shows that there is a correspondence between the equilibria of (i) the game $\mathcal{A}(\Gamma)$ where each player i observes some private information s_i and (ii) the game $\mathcal{S}(\Gamma)$, where all players observe some public information θ and have no private information. Therefore this result provides a deeper understanding of certain strategic situations, where the equilibrium outcomes are the same when (i) each player i observes private information s_i and (ii) players all observe the same piece of public information θ.

1.9 Applications: Symmetric information and asymmetric information

To illustrate the relevance of the link between $\mathcal{A}(\Gamma)$ and $\mathcal{S}(\Gamma)$, we now present two applications. The first application focuses on second price auctions, while the second application studies creditors bargaining in a bankruptcy situation. We then provide two examples to show that the equivalence of asymmetric and symmetric games is indeed important.

1.9.1 Second Price Auctions

First we consider a second price auction where players have valuations s_i and there is an unknown reserve price θ. In the first situation, there are two collectors interested in buying a first edition book. They are labeled $\{1, 2\}$. It could be the case that each collector knows how much he values the book (ie the value of s_i) but does not know the reserve price set by the seller (ie the value of θ). Collectors may then choose to enter the auction with $a_i = E$ or choose not to enter the auction with $a_i = NE$. To order the decisions, we assign $NE = 0$ and $E = 1$. Each collector who enters submits a secret bid (a collector who does not enter submits a bid of 0), and if the reserve price is not met then there is no sale. Crucially there is a cost to attending the auction given by c. Therefore a collector may be put off attending the auction because of the cost involved in participating. Assuming that when a collector chooses to enter he bids his valuation, the following utility function represents this situation.

$$u_i(\mathbf{a}, \theta, \mathbf{s}) = \begin{cases} s_i - \max\{s_j, \theta\} - c & \text{if} \quad a_i = P \qquad a_j = P \qquad s_i > \max\{s_j, \theta\} \\ -c & \text{if} \quad a_i = E \quad a_j = \{E, NE\} \quad s_i < \max\{s_j, \theta\} \\ 0 & \text{if} \quad a_i = NE \end{cases} \tag{1.10}$$

Secondly we consider a second price auction for oil tracts. Now consider a situation with two oil firms labeled $\{1, 2\}$. It could well be the case that the buyer knows the reserve price (denoted by θ), but does not can only estimate how much oil the tract contains and hence the value of the oil tract (denoted by s_i). Firms may then choose to enter the auction with $a_i = E$ or choose not to enter the auction with $a_i = NE$. Each firm who enters submits a secret bid (a firm who does not enter submits a bid of 0), and if the reserve price is not met then there is no sale. Again there is a cost to attending the auction of c. Therefore as before a firm may be put off attending the auction because of the cost involved in participating. This situation is represented by exactly the same utility function as above, but instead of observing s_i firms observe θ.

Note that the utility function is the same in both cases and players are risk neutral expected utility maximisers. Assuming that in both situations $g(\theta) = 1$ for all $\theta \in \mathbb{R}$ ensures assumption 1 holds. In addition assuming that the distribution of valuations such that definition 3.5.1 is satisfied, we can apply theorem 1.8.1. From the theorem it follows directly, that the two games described have the same set of constant equilibria.

Since the second game is a game of symmetric information, it is possible to average over the uncertainty to form a complete information game $\mathcal{C}(\Gamma)$. We define

$$\pi_i = P(s_i > \max\{s_j, \theta\})E[s_i - \max\{s_j, \theta\}|s_i > \max\{s_j, \theta\}]$$

to be the expected payoff of player i given that both players participate in the auction:

	E	NE
E	$\left(\pi_1 - c, \pi_2 - c\right)$	$\left(\frac{E[s_1-\theta\|s_1>\theta]}{P(s_1>\theta)} - c, 0\right)$
NE	$\left(0, \frac{E[s_2-\theta\|s_2>\theta]}{P(s_2>\theta)} - c\right)$	$\left(0, 0\right)$

Having tackled the problem using a general distribution, to fix ideas we now consider a specific example. Say s_1 is drawn uniformly from $[\theta, \theta + 6]$, while s_2 is drawn uniformly form $[\theta, \theta + 4]$. The table above now reduces to:

	E	NE
E	$(2 - c, \frac{2}{3} - c)$	$(3 - c, 0)$
NE	$(0, 2 - c)$	$(0, 0)$

If $c \leq \frac{2}{3}$, then it is a dominant strategy for each player to enter the auction. This is because player 1 receives (at worst) an expected payoff of $2 - c > 0$, while player 2 receives (at worst) an expected payoff of $\frac{2}{3} - c \geq 0$.[7] Hence an auctioneer can

[7]Despite the weak inequality it is still a dominant strategy because if player 1 chooses NE

guarantee himself revenue $R = \min\{s_1, s_2\} + \frac{4}{3}$ by setting the entry cost c to be $\frac{2}{3}$. This is better than simply running a second price auction where the auctioneer raises revenue $\min\{s_1, s_2\}$.

If $c \in (\frac{2}{3}, 3)$ only player 1 will participate in the auction. Hence the auctioneer will sell the object at the reserve price of $\theta \leq \min\{s_1, s_2\}$. Hence in this case the optimal entry cost is $\frac{2}{3}$. This analysis shows that an auctioneer can set an entry cost players are always willing to pay. Importantly the entry cost *does not jeopardise the chance that the object is sold*. This is true even though the reserve price is included in the support of all the players. This effect is driven by the fact that no player knows he has a valuation close to the reserve price and so each player is willing to pay an entry fee in the hope that he has a valuation significantly above the reserve price. This strikingly differs from the standard model, where players with low valuations are unwilling to pay entry fees.[8]

This simple example gives an indication of how the modelling tool proposed in our paper can be applied to second price auctions to help set either the entry cost or the reserve price. The next section looks at how this modelling tool can be used to uncover links between games which have been studied in the literature.

1.9.2 Bankruptcy and Bargaining

Consider a company going bankrupt. There are two senior creditors numbered $\{1, 2\}$. Creditor i is owed s_i. However there is only θ to distribute and it may be the case that $s_1 + s_2 > \theta$ and the company does not have enough money to fully repay its senior creditors.

Each creditor demands part of his money. Hence the strategy set for each player is given by $A_i = [0, 1]$, where $a_i = 1$ captures a creditor demanding all his money and $a_i < 1$ captures a creditor demanding only some of his money.

If there the company has enough money to satisfy both demands then each creditor is paid the amount he demanded (any surplus is divided between junior creditors).

then $2 - c > 0$

[8] A resulting effect of the standard model is that entry fees typically mean that the object may not be sold.

However if the company does not have enough money to satisfy both demands then creditors enter arbitration. Each creditor is awarded a fraction β_i of the surplus. However since $\beta_1 + \beta_2 < 1$, there is always an agreement that Pareto dominates disagreement.

$$u_i(\mathbf{a}, \theta, \mathbf{s}) = \begin{cases} a_i s_i & \text{if} \quad a_1 s_1 + a_2 s_2 \leq \theta \\ \beta_i s_i & \text{otherwise} \end{cases} \tag{1.11}$$

We assume a scalable information structure with $G(\theta) = \ln(\theta)$ and $g(\theta) = \frac{1}{\theta}$. Moreover, it can be checked that the preference relation is scalable with respect to this G satisfying assumption 1 and hence theorem 1.8.1 applies.

One typical situation in a bankruptcy case is where the assets of the company are unknown and players have private information about how much they are owed. This would be captured by the game $\mathcal{A}(\Gamma)$. Another situation is where the assets of the company are known, but players do not know exactly how much they will gain in arbitration. In case players choose which proportion of their claim to demand, this would be captured by the game $\mathcal{S}(\Gamma)$. This correspondence unifies much of the literature on bargaining.

1.10 Conclusion

In this paper we proposed a framework for modelling situations of asymmetric information. This framework can be used to model such situations in a tractable way and establishes a close connection between certain games of asymmetric information and games of symmetric information. The relevance of both points is illustrated using examples.

In future work we would like to use this framework to study particular situations in more detail and provide additional key applications. Such applications consider the risk estimates banks provide to the regulator as well as a general model for the formation of asset price bubbles. Furthermore an extension to multi dimensional signal spaces and action spaces may increase the relevance of this framework. In addition we are interested in comparing the scalable information structure proposed here, to other information structures typically used in the literature. In particular we would like to draw the link with the independent types assumption and consider intermediate cases. The goal is to develop a full comprehensive framework covering a range of information structures, for the general class of payoffs considered in this paper.

1.11 Appendix A: Proofs Analysis: Simplicity

1.11.1 Proof of proposition 1.6.1

Proof. Suppose the strategy profile σ^* is a Nash equilibrium of $\mathcal{C}(\Gamma)$. If $a_i^* \in \text{supp}(\sigma_i^*)$, then $(a_i^*, \sigma_{-i}^*) \succeq_i^{\mathcal{C}} (\hat{a}_i, \sigma_{-i}^*)$ for all $\hat{a}_i$.

Hence $L_i[a_i^*, \sigma_{-i}^*, 0_H] \succeq_i^{\mathcal{C}} {}_i L_i[\hat{a}_i, \sigma_{-i}^*, 0_H]$.

By the scalability of payoffs (definition 3.5.1) and the scalability of the information structure (definition 4):

$$L_i[a_i^*, \sigma_{-i}^*, 0_H] \succeq_i^{\mathcal{C}} L_i[\hat{a}_i, \sigma_{-i}^*, 0_H] \Rightarrow L_i[a_i^*, \sigma_{-i}^*, s_i^*] \succeq^{\mathcal{A}} L_i[\hat{a}_i, \sigma_{-i}^*, s_i^*]$$

Hence player i has no incentive to deviate from the strategy profile σ_i^* when he observes s_i^*.

□

1.12 Appendix B: Proofs Applications: Simplicity

1.12.1 Proof of proposition 1.7.1: Loss Aversion in an all pay contest

Proof.

$$V(a_i, a_{-i}|\beta) \quad = \quad \psi_i(a_i, a_{-i})s_i - as_i - \beta\Big[1 - \psi_i(a_i, a_{-i})\Big]\Big[\psi_i(a_i, a_{-i})s_i\Big]$$

Differentiating with respect to a_i gives:

$$\begin{aligned}\frac{\delta V}{\delta a_i} &= \frac{\delta\psi_i(a_i,a_{-i})}{\delta a_i} - 1 - \beta\Big[1-\psi_i(a_i,a_{-i})\Big]\Big[\frac{\delta\psi_i(a_i,a_{-i})}{\delta a_i}\Big] + \beta\Big[\frac{\delta\psi_i(a_i,a_{-i})}{\delta a_i}\Big]\Big[\psi_i(a_i,a_{-i})\Big] \\ &= \frac{\delta\psi_i(a_i,a_{-i})}{\delta a_i} - 1 - \beta\frac{\delta\psi_i(a_i,a_{-i})}{\delta a_i}\Big[1-\psi_i(a_i,a_{-i})\Big] + \beta\Big[\psi_i(a_i,a_{-i})\Big] \\ &= \frac{\delta\psi_i(a_i,a_{-i})}{\delta a_i} - 1 + \beta\frac{\delta\psi_i(a_i,a_{-i})}{\delta a_i}\Big[2\psi_i(a_i,a_{-i}) - 1\Big]\end{aligned}$$

$$\frac{\delta V}{\delta a_i} = \frac{\delta\psi_i(a_i,a_{-i})}{\delta a_i} - 1 + \beta\frac{\delta\psi_i(a_i,a_{-i})}{\delta a_i}\Big[2\psi_i(a_i,a_{-i}) - 1\Big]$$

Differentiating with respect to β yields:

$$\frac{\delta V}{\delta\beta\delta a_i} = 2\frac{\delta\psi_i(a_i,a_{-i})}{\delta a_i}\Big[\psi_i(a_i,a_{-i}) - \frac{1}{2}\Big]$$

Now it is assumed that V is a single-peaked function and we use the condition that $\psi_i(a_i,a_{-i})$ is strictly increasing in a_i. Hence at equilibrium $\frac{\delta V}{\delta a_i} = 0$. □

1.12.2 Proof of proposition 1.7.2: Loss aversion in a first price contest

Proof.

$$\begin{aligned}V(a_i,a_j|\beta) &= \psi_i(a_i,a_j)\Big[s_i - as_i\Big] - \beta\Big[1-\psi_i(a_i,a_j)\Big]\Big[\psi_i(a_i,a_j)(s-as_i)\Big] \\ &= \Big[(1-a_i)\psi_i(a_i,a_j)\Big]\Big[1-\beta\psi_i(a_i,a_j)\Big]\end{aligned}$$

$$\frac{\delta V}{\delta a_i} = \Big[(1-a_i)\psi_i(a_i,a_j)\Big]\Big[-\beta\frac{\delta\psi_i(a_i,a_j)}{\delta a_i}\Big] + \Big[(1-a_i)\frac{\delta\psi_i(a_i,a_j)}{\delta a_i} - \psi_i(a_i,a_j)\Big]\Big[1-\beta\psi_i(a_i,a_j)\Big]$$

Looking at just the terms involving β:

$$\begin{aligned}\frac{\delta V'}{\delta a_i} &= -\beta\Big[\frac{\delta\psi_i(a_i,a_j)}{\delta a_i}\Big((1-a_i)\psi_i(a_i,a_j)\Big) + \psi_i(a_i,a_j)\Big((1-a_i)\frac{\delta\psi_i(a_i,a_j)}{\delta a_i} - \psi_i(a_i,a_j)\Big)\Big] \\ &= -\beta\Big[\psi_i(a_i,a_j)^2\Big]\end{aligned}$$

Looking at terms not involving β:

$$\frac{\delta V''}{\delta a_i} = \Big[(1-a_i)\frac{\delta\psi_i(a_i,a_j)}{\delta a_i} - \psi_i(a_i,a_j)\Big]$$

In equilibrium $\frac{\delta V}{\delta a_i} = \frac{\delta V'}{\delta a_i} + \frac{\delta V''}{\delta a_i} = 0$. Hence:

$$0 = \Big[(1-a_i)\frac{\delta\psi_i(a_i,a_j)}{\delta a_i} - \psi_i(a_i,a_j)\Big] + \beta\Big[\psi_i(a_i,a_j)^2\Big]$$

Collecting terms:

$$\frac{\beta\Big[\psi_i(a_i,a_j)^2\Big]\Big] + \psi_i(a_i,a_j)}{\frac{\delta\psi_i(a_i,a_j)}{\delta a_i}} = \Big[(1-a_i)\Big]$$

$$a_i = 1 - \frac{\beta\Big[\psi_i(a_i,a_j)^2\Big]\Big] + \psi_i(a_i,a_j)}{\frac{\delta\psi_i(a_i,a_j)}{\delta a_i}}$$

$$a_i = 1 - \psi_i(a_i,a_j)\left[\frac{\delta\psi_i(a_i,a_j)}{\delta a_i}\right]^{-1}\Big[\beta\psi_i(a_i,a_j)+1\Big]$$

An increase in β leads to an increase in the right hand side of the equation (if a_i is held constant). Hence for the FOC to continue to hold a_i must increase.

□

1.13 Appendix C: Proofs Analysis: Equivalence

1.13.1 Proof of Theorem 1.8.1

Proof. We first prove this for the case where $f(s_i|\theta)$ is a discrete distribution.

Suppose $\sigma(\mathbf{a}|\mathbf{s}) = \sigma^C(\mathbf{a})$ is a constant strategy profile. Suppose also that σ is a pure strategy so that for some $\mathbf{a}^* = (a_1^*, ..., a_n^*)$, it holds that $\sigma^C(\mathbf{a}) = 1$ whenever $a_i \geq a_i^*$ for all $i \in I$ and $\sigma^C(\mathbf{a}) = 0$ otherwise. Suppose further that σ is a BNE of $A(\Gamma)$. This means that when player i has signal 0_H he has no incentive to deviate. Hence for all deviations $\hat{a}_i \in A_i$ it holds that $V_1(a_i^*, a_{-i}^*; 0_H) \geq V_1(\hat{a}, a_{-i}^*; 0_H)$ where:

$$V_1^A(a_i, a_{-i}) = \sum_{(\theta, s_{-i})} g(\theta|0_H) f_{-i}(\mathbf{s}_{-i}|\theta) u_i(a_i, a_{-i}; 0_H, \mathbf{s}_{-i}; \theta)$$

Note that:

$$g(\theta|0_H) = \frac{g(\theta)f(0_H|\theta)}{\sum_{\tilde{\theta}} g(\tilde{\theta})f(0_H|\tilde{\theta})}$$

Now define:

$$V_2^A(a_i, a_{-i}) = \sum_{(\theta, s_{-i})} g(\theta) f(0_H|\theta) f_{-i}(\mathbf{s}_{-i}|\theta) u_i(a_i, a_{-i}; 0_H, \mathbf{s}_{-i}; \theta)$$

Substituting $g(\theta|0_H) = \Big[g(\theta)f(0_H|\theta)\Big]\Big[\int g(\tilde{\theta})f(0_H|\tilde{\theta})d\tilde{\theta}\Big]^{-1}$ and multiplying each side by the constant in the second set of square brackets it follows that $V_2(a_i^*, a_{-i}^*; 0_H) \geq V_2(\hat{a}, a_{-i}^*; 0_H)$. Now define:

$$V_3^A(a_i, a_{-i}) = \sum_{(\theta, s_{-i})} f(0_H|\theta) f_{-i}(\mathbf{s}_{-i}|\theta) u_i(a_i, a_{-i}; 0_H \ominus \theta, \mathbf{s}_{-i} \ominus \theta; 0_H)$$

Note that by the extra condition imposed it follows that:

$$g(\theta) u_i(a_i, a_{-i}; \theta; 0_H, \mathbf{s}_{-i}) = g(0_H) u_i(a_i, a_{-i}; 0_H \ominus \theta, \mathbf{s}_{-i} \ominus \theta; 0_H)$$

It follows from this equation that $V_2^A(a_i, a_{-i}) = g(0_H) V_3^S(a_i, a_{-i})$ and hence $V_3^S(a_i^*, a_{-i}^*) \geq V_3^S(\hat{a}_i, a_{-i}^*)$.
Define also:

$$V_4^S(a_i, a_{-i}) = \sum_{(\hat{s}_i, \hat{s}_{-i})} f(\hat{s}_i|0_H) f_{-i}(\hat{\mathbf{s}}_{-i}|0_H) u_i(a_i, a_{-i}; \hat{s}_i, \hat{\mathbf{s}}_{-i}; 0_H)$$

Define $\hat{s}_i = 0_H \ominus_H \theta$ and $\hat{s}_j = s_j \ominus_H \theta$. Note that from the assumption that the distribution is scalable it follows that (i) $f(\hat{s}_i|0_H) = f(0_H|\theta)$ and (ii) $f(\hat{s}_i|0_H) = f(s_j|\theta)$. Using these facts and substitutions it follows that $V_3^S(a_i, a_{-i}) = V_4^S(a_i, a_{-i})$. Hence $V_4^S(a_i^*, a_{-i}^*) \geq V_4^S(\hat{a}_i, a_{-i}^*)$. This shows that $\sigma^C(\mathbf{a})$ is also a Nash equilibrium of the game of symmetric information. The reverse direction can easily be seen by repeating the steps above. Finally it is clear that the case where $f(s_i|\theta)$ is a continuous distribution (although needing more notation) can be proved along similar lines.

□

1.14 Appendix D: Additional Material

1.14.1 Alternative specification of preferences: scalable actions

We show that our framework can also be used to model situations of asymmetric information, where preferences over lotteries are unchanged when scaling the actions of all players, the signals of all players and the state. For instance, consider the case of an auction, where payoffs are homogeneous of degree one: multiplying the valuations, the bids of all players by a constant their payoffs are multiplied by the same constant. If the game satisfies this alternative definition of scalable preferences and the information structure is scalable, these games are strategically equivalent to a scalable game of asymmetric information $\mathcal{A}(\Gamma)$, where $a_i = \hat{a}_i \ominus_H s_i$ and $\hat{a}_i$ is the action chosen in this alternative game. These strategies are of the form $\sigma_i(s_i) = e_i^* \oplus_H s_i$, where e_i^* is a player specific constant. These games can be studied in their original form. However the translation to the scalable game form is useful for studying the link between asymmetric information and symmetric information.

Consider a game of asymmetric information, $\mathcal{A}(\Gamma) = I, (\hat{A}_i)_{i\in I}, (\succeq_i)_{i\in I}, g, (F_i)_{i\in I}$, where $\hat{A}_i = \Theta$ for all $i \in I$ and the information structure is scalable with respect to G (see definition 3.5.1). If the game $\mathcal{A}(\Gamma) = I, (A_i)_{i\in I}, (\succeq_i)_{i\in I}, g, (F_i)_{i\in I}$

Given $\omega = (\hat{\mathbf{a}}, \theta, \mathbf{s})$, let $\omega \hat{\oplus}_G k \equiv (\mathbf{a}\oplus_H, \theta \oplus_H k, \mathbf{s} \oplus_H k)$ and let $[L\hat{\oplus}_G k](w) \equiv L(w\hat{\oplus}_G k)$. Suppose the preference relations satisfy the following definition:

Definition 8 (Alternative scalable preference relations). *A preference relation $\succeq_i$ is alternatively scalable with respect to G if whenever:*

$$L \succeq_i L'$$

then,

$$[L\hat{\oplus}_H k] \succeq_i [L'\hat{\oplus}_H k]$$

This definition says that if a player prefers lottery L to lottery L' then, when all the actions, the state and the signals are scaled up by a constant, he continues to prefer the scaled up lottery arising from L to the one arising from L'. This definition differs from the standard definition of a scalable preference structure in that all the elements of ω are scaled including the actions.

In some applications, the preference structure satisfies this alternative definition of scalable preference relations. In these cases, there exists a transformation $a_i = \hat{a}_i \ominus_H$, which redefines the action space from $\hat{A}_i = \Theta$ to A_i, such that when the actions are changed from $\hat{a}_i$ to a_i, the transformed preference relations satisfy definition 4. Formally, this can be stated as follows:

Proposition 1.14.1. *In a game $\hat{\Gamma}^{\mathcal{A}} = \{I, (\hat{A}_i)_{i \in I}, (\succeq_i)_{i \in I}, g, F\}$, if $A_i = \Theta$ for all $i \in I$ and players' preferences satisfy definition 8, there exists a strategically equivalent game $\mathcal{A}(\Gamma) = \{I, (A_i)_{i \in I}, (\succeq_i)_{i \in I}, g, F\}$: If $\sigma_i^{\mathcal{A}}$ is an equilibrium of $\mathcal{A}(\Gamma)$, then $\hat{\sigma}_i^{\mathcal{A}}$ where $\hat{a}_i = a_i \oplus_H s_i$, is an equilibrium of $\hat{\Gamma}^{\mathcal{A}}$.*

The proof follows immediately from substitutions.[9]

As an example consider the case of an auction. The true actions players take are their bids $\hat{a}_i$ in $[0, \infty)$. A first price auction clearly satisfies definition 8. However instead of considering the bids directly, we can consider the case where players choose the proportion of their valuation they want to bid $a_i = \hat{a}_i \ominus_H s_i$. Using these proportions to describe a player's preferences, these satisfy definition 3.5.1.

[9] A similar result exists for $\mathcal{S}(\Gamma)$, but it is slightly more complicated, because actions require scaling by s_i which is not observed in $\mathcal{S}(\Gamma)$.

Chapter 2

Robustness of Subgame Perfect Implementation

2.1 Introduction

This paper studies the robustness of implementation in subgame perfect equilibrium (SPE) in the fashion of Moore and Repullo (1988) and Aghion et al. (2012).

A social choice function (SCF) is said to be implemented fully, if there exists a mechanism such that the outcome prescribed by the SCF is the unique equilibrium of the mechanism in all states. Subgame perfect implementation is relevant when sequential mechanisms are used. Although the existing literature on implementation in SPE characterizes the set of SCFs which can be implemented under different informational assumptions, these papers do not provide a distinction between SCFs that are seen to be implemented in practice and those that are not. This distinction is an important aim of implementation, as in any situation it allows a social planner to fully understand the set of SCFs he can choose from.

In this paper we show that placing a very reasonable restriction on the information players have about their own preferences and on the information they have about the preferences of others, allows to distinguish between SCFs which we are seen to be implemented in practice and those that do not appear. More precisely we focus on environments where information is almost complete and introduce information perturbations where each player has more precise information about his own preferences than do other players. These perturbations are referred to as *restricted information perturbations*.

Moore and Repullo (1988) show that under complete information almost any SCF can be implemented in SPE. Taking a step away from implementation under complete information, Aghion et al. (2012) (henceforward AFHKT) show that any implementation of a non-Maskin monotonic SCF is not robust to a general class of information perturbations we refer to as *full perturbations*. Maskin monotonicity is a very restrictive requirement and is violated by many SCFs that are implemented in practice, for example firms paying a higher wage to workers with higher outside options. The result obtained by AFHKT therefore questions the usefulness of subgame perfect implementation.

In this paper we argue that typically each player is better informed about his own preferences than is any other player. We restrict attention to a class of perturbations by requiring that players know their own preferences with certainty. This is a reasonable restriction as there are many situations where each player knows his preferences, while others may be slightly uncertain.[1] One example of such a setting is that studied by Bester and Kraehmer (2012) who consider a seller making an offer to a buyer who has private information about how much he values the good.

We show that these restrictions provide a good distinction between SCFs seen to be implemented in practice and those that are not. In particular we demonstrate that under these restricted information perturbations, a wide range of SCFs can be robustly implemented, including many that are not Maskin monotonic. The class of SCFs that can be implemented robustly under the *restricted perturbations* considered here is therefore strictly larger than those that can be implemented robustly under a wider range of full perturbations.

Informally, the reason why the implementability of certain SCFs is robust to restricted perturbations but not full perturbations is the following: Under restricted

[1]Our logic also applies to cases where a player is slightly uncertain about his own preferences, as long as he is more certain about them than is any other player.

perturbations, players know their preferences with certainty and do not gain information about their own preferences from the actions of another player. Meanwhile when using full perturbations players have some uncertainty about their own preferences, and hence may update their beliefs about their own preferences after the moves of other players. In particular, in a two-stage game the result of AFHKT relies on off-equilibrium beliefs which ensure that the second-mover gains a significant amount of information about his own preferences after observing an off-equilibrium move from the first-mover. The lack of belief updating considered here leads to a much larger class of robust mechanisms under restricted perturbations.

Consider the example of a single firm and two types of workers, who differ in their outside option. A 'bad' sequential equilibrium is one where a high type worker accepts a wage that is below his outside option. These equilibria may arise under full information perturbations and rely on the fact that the worker is less informed about his own preferences than the firm. This may occasionally be the case for example when the firm has more information about the job description than the worker. However in most situations this is unlikely to hold, for instance when the worker is more informed of his preferences or outside options. Hence in many applications *restricted perturbations* are the more appropriate tool for assessing whether a certain mechanism is robust. Using this analysis, subgame perfect implementation is very robust in settings where players are confident about which allocations they value.

For most of the paper, we restrict attention to non-stochastic mechanisms where players move sequentially. This restriction is motivated by the fact that in many situations mechanisms where players move simultaneously are not feasible. For instance when bargaining a player must observe the offer made by his opponent before deciding whether to accept or reject the offer made: indeed in most bargaining models - for instance Rubinstein (1982) - players move sequentially. In contrast Baliga (1999) and Bergin and Sen (1998) study implementation in a similar setting with incomplete information and extensive form games, but where players choose their actions simultaneously. These papers show that allowing players to move simultaneously leads to much more permissive results than those presented here.

Meanwhile Corchón and Ortuno-Ortín (1995) and in a generalisation Yamato (1994) consider similar information structures where each player perfectly knows the preferences of other players in his own group but has imperfect information about players outside his group. Using Bayesian and dominant strategy implementation as equilibrium concepts they find that Nash implementation in complete information is a necessary and sufficient condition for robust implementation. In this paper we focus on a two player setting and study subgame perfect implementation which is particularly relevant in sequential move games.

Our main result relates to the concept of *exact implementation* studied by Moore and Repullo (1988) as well as Abreu and Matsushima (1994). The term exact implementation in a setting with information perturbations is used to mean that the desired allocation is always implemented whenever players observe correct signals about the state. The main result of our paper then proves a sufficient condition for a SCF to be exactly implementable with restricted information perturbations. In particular we show that any SCF which can be implemented in a two-stage sequential move game in complete information can be implemented exactly with restricted information perturbations. Moreover requiring two stage implementation is more permissive than requiring Maskin monotonicity, but more restrictive than requiring only three stage implementation.

Since the necessary and sufficient conditions for two stage implementation do not provide great insight, the relevance of two stage implementation is illustrated using a number of examples. Many standard settings of principal agent interaction proceed in two stages, where the principal offers a contract. The agent can reject the contract, accept it - or in some cases - choose an action. Indeed, the examples given in this paper can be interpreted as classic principal agent settings. More precisely, the analysis can be be interpreted as studying the robustness of the outcome of principal agent interactions to small levels of asymmetric information.

Finally, we consider the weaker concept of virtual implementation studied by Abreu and Sen (1991). Virtual implementation with information perturbations requires that the desired allocation is implemented almost always, but does not exclude the possibility for the wrong allocation to be occasionally implemented even when players observe the correct signals. In a deviation from most literature we do not consider virtual implementation using a stochastic element in the mechanism.[2] Instead we follow an approach introduced by Serrano and Vohra (2010) and allow players to choose mixed strategies. We say that an SCF is virtually implementable when in the only equilibrium of the game with information perturbations, players choose mixed strategies, such that the outcome prescribed by the SCF is reached almost always and the probability with which any type chooses a different path becomes arbitrarily small when the information perturbations tend to zero.

Using an example, we show that requiring only virtual implementation some SCFs are robust to restricted information perturbations, although they are not robust when exact implementation is required. This argument shows that the set of SCFs that can be considered robust to information perturbations become larger when considering weaker concepts of implementation. The decision of which concept is appropriate may depend on the situation one has in mind.

The remainder of the paper proceeds as follows. In section two we provide an example to illustrate the differences between implementability under complete information, full perturbations and restricted perturbations respectively, as well as present the intuition behind these differences. Section three introduces the model and formal definitions. The sufficient condition for robust implementation under restricted perturbations is presented in section four. In section five we consider the case of virtual implementation. Section six concludes.

2.2 Example

Suppose a firm (P) is bargaining with a worker (A). There are two states of the world $\Theta = \{L, H\}$, which represent the fact that workers may either be high type (H) or low type (L). The probability that the worker is high type is $\alpha_H \in (0, 1)$, while the probability that the worker is low type is $\alpha_L = 1 - \alpha_H$. There are three

[2] This approach is often criticised, because implementation relies on the mechanism designer committing to occasionally implement an allocation that he knows is not Pareto efficient at the point of implementing it.

outcomes $X = \{w_H, w_L, d\}$. First a high wage w_H may be agreed, secondly a low wage w_L may be agreed and thirdly a default option d may be reached. Both types of workers are equally productive when working for the firm and so the preferences of the firm do not depend on the type of the worker. The firm prefers to pay a low wage rather than a high wage, and prefers to pay a high wage rather than failing to make an agreement:

$$\text{Firm's preferences:} \quad u_P(w_L;\theta) > u_P(w_H;\theta) > u_P(d;\theta) \quad \text{for } \theta \in \{L, H\}$$

Meanwhile, all workers prefer the high wage to any other alternative. However, low type workers prefer to receive the low wage rather than the outside option, while the high type workers prefer the outside option to the low wage. Therefore the preferences of each type of worker are given as follows:

$$\begin{array}{lll} \text{Low type's preferences:} & u_A(w_H;\theta) > u_A(w_L;\theta) > u_A(d;\theta) & \text{for } \theta = L \\ \text{High type's preferences:} & u_A(w_H;\theta) > u_A(d;\theta) > u_A(w_L;\theta) & \text{for } \theta = H \end{array}$$

All of the above is commonly known. Players negotiate according to the following two-stage sequential move bargaining procedure. In the first stage the firm makes an offer $w \in \{w_L, w_H\}$, and then in the section stage the worker chooses to accept (Y) or decline (N) the offer. If the worker accepts the wage offer this agreement is made, and otherwise the default option is reached. The extensive-form version of this game is given in Figure 1.[3]

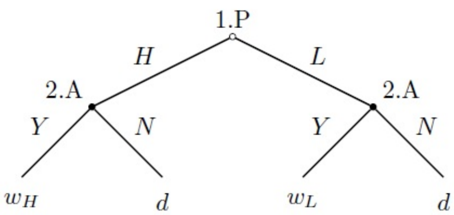


Figure 2.1. Two stage mechanism.

We analyse this game under three different information structures. In the first case

[3]Each node is an information sets and there are no moves by nature, as we assume that workers are born with their preferences.

we consider complete information, where both players know the worker's type. In the second and third case, we assume that one player knows the worker's type, while the other receives a signal $s \in \{s_L, s_H\}$ which is highly correlated with the worker's type. More precisely after observing a signal s_L the probability of the worker being low type is equal to $(1-\epsilon)$, while after observing a signal s_H the probability of the worker being high type is equal to $(1-\epsilon)$. After receiving such a signal a player is highly confident - although not completely sure - about the worker's type: in this case we say the worker's type is ϵ-known. Throughout the example it is assumed that $\epsilon > 0$ and ϵ is sufficiently small. A more formal approach is taken in the next section.

Complete information

First consider the case of complete information, where the worker's type is commonly known. In this case there is a unique SPE, where on the equilibrium path the firm offers the low type worker the low wage, the firm offers the high type worker the high wage and all offers are accepted. Off the equilibrium path, low type workers accept a high wage and high type workers reject a low wage. Therefore in complete information this mechanism implements a SCF $f(\theta)$, where $f(L) = w_L$ and $f(H) = w_H$. Note that this SCF is not Maskin monotonic, since both types of workers prefer a high wage to a low wage and yet only the high type workers receive a high wage while the low type workers receive a low wage. Formally Maskin monotonicity is defined as follows:

Definition 9 (Maskin monotonicity). *An SCF ψ is Maskin monotonic, if for all $\theta, \theta' \in \Theta$:*

$\psi(\theta) = x$ and $\theta' \in L_i(x, \theta_i)$ for $i = A, B$ imply $\psi(\theta') = x$

where $L_i(x, \theta_i)$ is the lower contour set of player i with preferences θ_i at allocation x.

An information perturbation where workers know their own preferences

Secondly consider the case where the worker's type is known by the worker and ϵ-known by the firm. Since the worker knows his own type, high type workers always reject the low wage, while low type workers always accept it. Given ϵ is sufficiently small it follows that:

$$\alpha_L(1-\epsilon)\Big(u_P(w_L;\theta)-u_P(w_H;\theta)\Big) > \alpha_H\epsilon\Big(u_P(w_H;\theta)-u_P(d;\theta)\Big)$$

The left hand side represents the firm's gains when offering a low wage rather than a high wage to a low type player having received a signal s_L which was correct. Meanwhile the right hand side denotes the losses that the firm incurs when offering a low wage - which is rejected - rather than a high wage after an incorrect signal s_L. If ϵ is sufficiently small and the signal is sufficiently reliable, it is clear that the gains from offering a low wage outweigh the loss of occasionally reaching the default after an incorrect signal. It follows that there is a unique sequential equilibrium where the firm offers a low wage after observing a signal s_L and a high wage after observing a signal s_H. Note that the unique sequential equilibrium is very close to the complete information SPE. Hence this mechanism can be considered robust to those information perturbations where the worker knows his own preferences.

An information perturbation where workers do not know their own preferences

Finally, consider the case where the worker's type is known by the firm and ϵ-known by the worker. In this case there are two distinct sequential equilibria. First there is a separating equilibrium, which is almost outcome-equivalent to the complete information SPE. In the first stage the firm nearly always offers a high type worker the high wage and a low type worker the low wage. Then in the second stage the workers always accept if they receive a high wage or if they receive a low wage and have a low signal. They play mixed strategies when the firm offers a low wage and they receive a high signal. If ϵ is small this third case happens rarely, and the complete information outcome is nearly always reached. In this 'trusting' sequential equilibrium, workers believe the firm is very likely to have made the appropriate offer unless they have reason to believe otherwise.

However, there is also another pooling equilibrium which leads to a very different outcome. In the first stage the firm offers all workers the high wage, and in the second stage all workers accept. To ensure that this is indeed a sequential equilibrium, it is assumed that workers have the following off-equilibrium beliefs: if the firm makes a low offer (which does not happen in equilibrium), then the worker believes he is very likely to be a high type regardless of his initial signal. This means that the off-equilibrium beliefs are such that the firm's off-equilibrium move is *much more informative than the worker's original signal.* Therefore when a worker who has received a low signal s_L receives a low offer w_L, he believes there is a significant chance that he is high type and rejects the offer. In this 'suspicious' pooling equilibrium workers do not believe that the firm has made the appropriate offer when the firm makes an off-equilibrium move. These suspicious off-equilibrium beliefs sustain what AFHKT refer to as a 'bad' sequential equilibrium.

AFHKT prove that any mechanism implementing a non-Maskin monotonic SCF in complete information is not robust to certain information perturbations. This example suggests that this result relies on the fact that players learn about their own preferences from the actions of other players. The main result of this paper formalises this. We show that bad sequential equilibria arise precisely in the case where the second mover significantly updates his belief about his own preferences from observing the other player's move. We prove that any SPE implementation in complete information which uses a two stage sequential mechanism is robust to those information perturbations where players remain certain of their own preferences. This shows that many SPE implementations in complete information are robust to the class of perturbations which are most relevant for many situations.

2.3 The model

There are two players $i = \{A, B\}$ and the payoff type of each player is denoted by $\theta_i \in \Theta_i$. The state is given by the pair of payoff types $\theta = (\theta_A, \theta_B) \in \Theta_A \times \Theta_B = \Theta$. We let X denote the set of allocations, while players' Bernoulli utilities are denoted by $u_i(x; \theta_i)$. These utilities depend only on the eventual allocation $x \in X$ and the player's type θ_i. It is assumed that the state space Θ and the set of outcomes X are finite. A complete information SCF f is a one to one mapping from a state to an outcome, $f: \Theta \mapsto X$.

Before any move is made, player A observes a signal $s^A = (s^A_A, s^A_B) \in S^A$ and player B observes a signal $s^B = (s^B_A, s^B_B) \in S^B$ where s^i_j is a signal about player $j's$ preferences. We identify the signal sets with the state space so that $S^A = S^B = \Theta$. Signals are drawn from a common prior described by $\nu \in V$, where $\nu : \Theta \times S^A \times S^B \mapsto [0,1]$ and $\sum \nu = 1$.

We restrict our focus to extensive form mechanisms Γ with a finite number of stages where players move sequentially and every move is immediately and perfectly observed by the other player. Without loss of generality it is assumed that player A moves first, players move alternately and the number of stages is $2N$ for some $N \in \mathbb{N}$).

In any stage n, if n is odd then player A chooses a strategy $\sigma_{A,n} \in \Sigma_{A,n}$, while if n is even then player B chooses a strategy $\sigma_{B,n} \in \Sigma_{B,n}$. Therefore in the first stage player A makes a move, in the second stage player B moves and so on. Let $\sigma_A = (\sigma_{A,1}, \sigma_{A,3} .... \sigma_{A,2N-1})$ and $\sigma_B = (\sigma_{B,2}, \sigma_{B,4} .... \sigma_{B,2N})$ denote a possible set of strategies for player A and player B respectively. Furthermore let $\sigma = (\sigma_A, \sigma_B)$, and write $\Gamma(\sigma) \in X$ to mean the allocation implemented when players choose strategies σ. It is assumed that all strategy sets $\Sigma_{A,n}$ and $\Sigma_{B,n}$ are finite.

Players may condition their strategies on their signal and previously observed moves. Hence a strategy profile $h_{i,n}$ at stage n for player i maps a vector $(s^i, \sigma_{i,1}, \sigma_{i,2}, ..., \sigma_{i,n-1})$ to a strategy σ_n. A complete strategy profile h_i for player i denotes a set of strategy profiles for each stage where that player moves. Hence the strategy profile $h = (h_A, h_B)$ is a subgame perfect equilibrium (SPE) of the complete information game Γ if players have no incentive to deviate from this strategy profile.

Players initially form their beliefs based on their signal and the initial common prior. As the game progresses, players may update their beliefs after the move of an opponent. A belief profile $\phi_{i,n}$ for player i at stage n maps a vector $(s^i, \sigma_{i,1}, \sigma_{i,2}, ..., \sigma_{i,n-1})$ to a prior ν. A complete belief profile ϕ_i denotes a set of belief profiles for every stage, and $\phi = (\phi_A, \phi_B)$ denotes a pair of such belief profiles. The pair (h, ϕ) is a sequential equilibrium (SE) induced by the game (Γ, v) if ϕ represents a set of consistent beliefs given that (i) players are playing according to the strategy profile h and (ii) given their beliefs ν players have no incentive to deviate from the strategy profile h in any information set.[4]

2.3.1 Three informational environments

We now outline three possible restrictions on the prior ν which capture three different informational environments. First consider a complete information environment where players are certain of each others preferences. This is only the case when players always receive the correct signal about their own preferences and the preferences of their opponent. Hence we say that ν^0 is a complete information prior if ν puts probability 1 on $s^A = s^B = \theta$.

Definition 10 (Complete information)**.** *The prior ν^0 is a complete information prior, if and only if*

$$\sum_{\theta\in\Theta} \nu^0(\theta,\theta,\theta) = 1$$

Secondly consider the environment where both players observe a highly reliable signal about the preferences of both players as studied by AFHKT. In particular suppose that the reliability of the signal is such that a player is misinformed about either the preferences of his opponent or his own preferences with a probability lower than ϵ. Therefore $s^A = \theta$ and $s^B = \theta$ with probability greater than $1 - 2\epsilon$, and hence we define a full (ϵ)-perturbation as follows:

Definition 11 (Full (ϵ)-perturbations)**.** *The prior ν^ϵ is a full (ϵ)-perturbation if and only if*

$$\sum_{\theta\in\Theta} \nu^\epsilon(\theta,\theta,\theta) > 1 - 2\epsilon$$

Finally consider an environment where players are certain of their own preferences and observe a highly reliable signal about the preferences of the other player. Suppose that players are misinformed about the preferences of his opponent with a probability lower than ϵ. As before, since players are almost always correctly informed about both their preferences and their opponent's preferences $s^A = \theta$ and $s^B = \theta$ with probability $1 - 2\epsilon$. However since players are certain of their own

[4]This definition follows Aghion et al. (2012) who provide a formal definition of a sequential equilibrium in these multistage games in their online appendix.

preferences there is an additional requirement, since both $s_A^A = \theta_A$ and $s_B^B = \theta_B$ with probability 1. Hence a prior ν^ϵ with restricted (ϵ)-perturbations is defined as follows:

Definition 12 (Restricted-(ϵ) perturbations)**.** *The prior ν^ϵ is a restricted (ϵ)-perturbation if and only if*

1. *ν^ϵ is a full (ϵ)-perturbation*
2. *If $s_A^A \neq \theta_A$, then $\nu^\epsilon(\theta, s^A, s^B) = 0$*
3. *If $s_B^B \neq \theta_B$, then $\nu^\epsilon(\theta, s^A, s^B) = 0$*

Finally define V_C to be the set of complete information priors, V_F^ϵ to be the set of full (ϵ)-perturbations and V_R^ϵ to be the set of restricted (ϵ)-perturbations. Note that $V_C \subset V_R^\epsilon \subset V_F^\epsilon$. The next two sections investigate under what conditions exact implementation and virtual implementation are robust to restricted (ϵ)-perturbations.

2.4 Exact implementation

We now give a definition of exact implementation in an environment with information perturbations. We say that a SCF f is robustly implementable with information perturbations if - when perturbations are sufficiently small - the desired outcome is implemented with probability one whenever players receive the correct signals.[5] Under information perturbations, the definition of exact implementation can be extended as follows:

Definition 13. *A mechanism Γ exactly implements a SCF $f : X \mapsto \Theta$ with restricted (full) perturbations if and only if given any complete information prior $\nu^0 \in V_C$ and any sequence of priors $\{\nu^\epsilon\}_{\epsilon>0}$ whenever*

1. *$\nu^\epsilon \in V_R^\epsilon$ ($\nu^\epsilon \in V_F^\epsilon$)*

[5]Note that the standard definition of exact implementation requires the desired allocation to be implemented with probability one in all cases. Under information perturbations this definition leads to trivial results, since clearly the wrong allocation will arise when players receive the wrong signals. For the analysis to be sensible, the definition is adapted to allow for other outcomes in the rare case, where players receive wrong signals.

2. *The sequential equilibrium* $(\sigma^\epsilon, \phi^\epsilon)$ *is induced by the game* (Γ, ν^ϵ)

then there exists some $\bar{\epsilon}$ *such that* $\Gamma(\sigma^\epsilon) = f(\theta)$ *whenever i)* $\epsilon < \bar{\epsilon}$ *and ii)* $s^A = s^B = \theta$

Using this definition, the main result of AFHKT applies in our setting:

Theorem 2.4.1 (AFHKT)**.** *An SCF* f *can be robustly implemented with full perturbations if and only if*

1. f *is Maskin-monotonic*

2. f *is implementable in a complete information setting*

This result holds in a very general setting with $n \geq 2$ players, where moves may be either sequential or simultaneous. It relies on the fact that in extensive form games with several stages, additional equilibria can be formed by choosing off-equilibrium beliefs judiciously. We discussed an example of an additional bad equilibrium that arises when full perturbations are considered in the previous section. It follows that using additional stages does not increase the number of SCFs that can be implemented. As shown by AFHKT, certain small information perturbations can reduce the power of sub-game perfect implementation significantly.

However if we rule out the possibility that players are mistaken about their own preferences and only consider this smaller class of restricted perturbations, the situation is not nearly so bleak. Our example has already shown that the implementability of some SCFs are robust to restricted perturbations and not full perturbations. We now generalise this result and give a sufficient condition for exact implementation under restricted perturbations.

2.4.1 Sufficient condition

In this section we introduce a sufficient condition for exact implementation with restricted information perturbations which is significantly weaker than Maskin-monotonicity. This shows that restricting the set of information perturbations in an intuitive way significantly increases the set of SCFs that are robustly implementable. We first make the following definition:

Definition 14 (F_2)**.** *An SCF $f \in F_2$ if it can be implemented under complete information by a two stage mechanism with sequential moves.*

We now state our sufficient condition for robust implementation with restricted information perturbations:

Theorem 2.4.2 (Sufficiency)**.** *If an SCF $f \in F_2$, then it can be robustly implemented with restricted information perturbations.*

In order to prove this result we first characterize the SPEs under full information in two stage sequential move games. We have to show that the strategy profile used in any sequential equilibrium with sufficiently small restricted information perturbations coincides with a SPE in complete information. It is easy to show that the second mover - assuming he receives the correct signal - chooses his strategy in the same way as he does under complete information, because when making his decision, the second mover has not updated his preferences and simply chooses the allocation he likes most. Given that the second-mover behaves as he does under complete information, it is then possible to show that the first-mover also behaves as he does under complete information as long as his signal is correct and perturbations are sufficiently small. The complete proof can be found in the appendix.

2.4.2 Comparison with complete information

In order to illustrate that restricted perturbations provide an appropriate criterion for distinguishing between SCFs which are seen to be implemented in practice and those that are not, we now provide a comparison with the case of complete information. We show that robustness to restricted perturbations is more restrictive than implementation under complete information.

We consider the canonical mechanism introduced by Moore and Repullo (1988). Although this mechanism can be used to implement a wide-range of SCFs under complete information, it is not robust to restricted perturbations. More precisely there exist SCFs which can be exactly implemented using this mechanism under complete information, but cannot exactly be implemented under restricted perturbations. Hence exact implementation under restricted perturbations is a more

Table 2.1. Example: Simple three stage mechanism: Implementable under complete information, not implementable under restricted perturbations

Preferences	
Firm: $\theta \in \{L, H\}$	$u_P(w_L;\theta) > u_P(x_H;\theta) > u_P(w_H;\theta) > u_P(x_L;\theta)$
Low type: $\theta = L$	$u_A(w_H;\theta) > u_A(w_L;\theta) > u_A(x_L;\theta) > u_A(x_H;\theta)$
High type: $\theta = H$	$u_A(w_H;\theta) > u_A(w_L;\theta) > u_A(x_H;\theta) > u_A(x_L;\theta)$

restrictive criterion than exact implementation under complete information. In particular many SCFs that require complex mechanisms to be implemented under complete information can not be implemented when allowing for restricted information perturbations.

This is illustrated using the following example. Again consider a setting where a firm denoted by P wants to hire a a worker denoted by A. The worker may be a high type or a low type. In this example there are two outside options denoted x_H and x_L respectively. The players' preferences are given in Table 1. Now consider the mechanism represented in Figure 2.

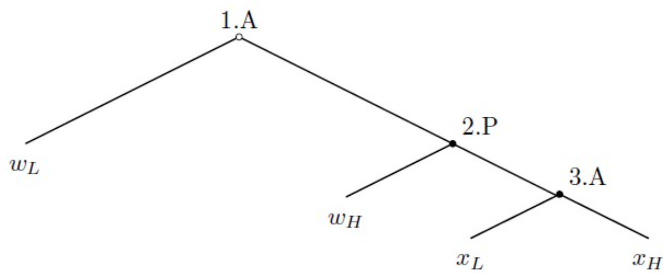


Figure 2.2. Moore and Repullo mechanism

Under complete information this Moore and Repullo mechanism implements the SCF where the high type worker receives w_L and the low type worker receives w_H. However note that the separating equilibrium implemented under complete information is not robust to restricted information perturbations. If the firm believes that it faces a high type whenever a worker starts by choosing the branch on the right, the firm will react by offering the worker the choice between the two outside options. This creates a 'bad' pooling sequential equilibrium in which all workers receive w_L. The SCF where the low type worker receives w_H and the high type worker receives w_L, is therefore not robust to restricted information perturbations. Hence this shows that the canonical Moore-Repullo mechanism is not robust to restricted perturbations.[6].

Other examples of two stage sequential move mechanisms seen in practice include a decision on a public good, where one agent announces how much he is willing to contribute, before a second agent decides to raise the amount to the critical threshold or to not contribute. Alternatively one can think of a principal agent setting, where the principal offers a menu of contracts and the agent chooses his preferred contract.

One should note that implementability in two stages under complete information is sufficient for exact implementation with restricted perturbations, but is not necessary. In the appendix we present an example of an SCF that can be exactly implemented in three stages with restricted perturbations but not in two stages.[7]

2.5 Virtual Implementation

In this section we show that the range of SCFs that are robust to restricted information perturbations becomes even larger, when considering the weaker concept of virtual implementation. Formally virtual implementation requires that for each $\epsilon > 0$ there exists a nearby game Γ^ϵ such that in any sequential equilibrium of this game the desired outcome is obtained with probability greater than $1 - \epsilon$. This is weaker than the concept of exact implementation considered previously,

[6] Note that this does not prove that the SCF cannot be implemented robustly. But it cannot be implemented robustly using the mechanism suggested by Moore and Repullo (1988)

[7] However, these examples are rare and difficult to construct. In particular the example we present is such that by allowing for simultaneous move in the first stage and then allowing one of the players to move again in the second stage, the SCF can be implemented in two stages. Hence by weakening condition $F2$ to implementability in two stages where the first stage allows for simultaneous moves, while only one player moves in the second stage.

since we now allow for the possibility that the desired outcome is occasionally not implemented even in cases when both players receive the correct signals. More precisely:

Definition 15. *A mechanism* Γ *virtually implements an SCF* $f : X \mapsto \Theta$ *with restricted (full) perturbations if and only if given any* $\delta > 0$*, any complete information prior* $\nu^0 \in V_C$ *and any sequence of priors* $\{\nu^\epsilon\}_{\epsilon>0}$ *whenever*

1. $\nu^\epsilon \in V_R^\epsilon$ *(*$\nu^\epsilon \in V_F^\epsilon$*)*
2. *The sequential equilibrium* $(\sigma^\epsilon, \phi^\epsilon)$ *is induced by the game* (Γ, ν^ϵ)

then there exists some $\bar{\epsilon}$ *such that* $P\Big(\Gamma(\sigma^\epsilon) = f(\theta)\Big) > 1 - \delta$ *whenever* $\epsilon < \bar{\epsilon}$

Most previous work on virtual implementation - see Serrano and Vohra (2010) for an exception - considers stochastic mechanisms where in equilibrium players play according to pure strategies. In these cases the slight uncertainty over the eventual outcome is caused by the stochasticity of the mechanism. In contrast, in the examples considered below slight uncertainty over the eventual outcome is caused by the fact that players do not play pure strategies, but rather play *almost pure* strategies, allowing them to deviate from the main strategy prescribed for their type occasionally.

Virtual implementation under restricted perturbations is less permissive than virtual implementation under complete information, while being more permissive than exact implementation under restricted perturbations. To show the first part of this claim it is sufficient to consider the canonical Moore-Repullo mechanism analysed above. It can immediately be seen that this mechanism - and hence the canonical mechanism - is not robust to restricted perturbations even when considering the weaker criterion of virtual implementation. This follows from the fact that this mechanism has a pooling equilibrium, as explained in the previous section. Whenever 'bad' sequential equilibria arise from pooling, both virtual implementation and exact implementation fail.

To show the second part of this claim we provide an example of an SCF which cannot be robustly implemented under restricted perturbations if exact implementation is required, but is robust when requiring only virtual implementation.

Table 2.2. Complex three stages: Virtually implementable under restricted perturbations not exactly implementable under restricted perturbations

Preferences	
Firm: $\theta \in \{L, H\}$	$u_P(y_L;\theta) > u_P(w_L;\theta) > u_P(x_H;\theta) > u_P(w_H;\theta) > u_P(x_L;\theta) > u_P(y_H;\theta)$
Low type: $\theta = L$	$u_A(w_H;\theta) > u_A(w_L;\theta) > u_A(x_L;\theta) > u_A(x_H;\theta) > u_A(y_L;\theta) > u_A(y_H;\theta)$
High type: $\theta = H$	$u_A(w_H;\theta) > u_A(w_L;\theta) > u_A(x_H;\theta) > u_A(x_L;\theta) > u_A(y_H;\theta) > u_A(y_L;\theta)$

This difference follows from the fact that exact implementation requires the complete information allocation to be implemented whenever both players receive the correct signal. Virtual implementation allows rare occasions where players deviate from their complete information strategy in which case a different allocation is implemented despite both players receiving the correct signal. Robust virtual implementation requires these 'differences' to become increasingly rare as signal precision increases. An example of such a setting is discussed below.

2.5.1 Comparison with exact implementation

We now give an example of an SCF which can be virtually implemented robustly, but cannot be exactly implemented robustly. Note also that the example is constructed such that the SCF can be virtually implemented robustly using a three stage mechanism, even though it cannot be virtually implemented using a two-stage mechanism.

Let $\Theta = \{L, H\}$, $X = \{w_L, w_H, x_H, x_L, y_H, y_L\}$ and consider the preference profile given in Table 2.

Now consider the SCF $f : \Theta \mapsto X$, where $f(H) = w_L$ and $f(L) = w_H$. This SCF is implementable using restricted perturbations but it is not implementable in a two stage sequential move mechanism in complete information.

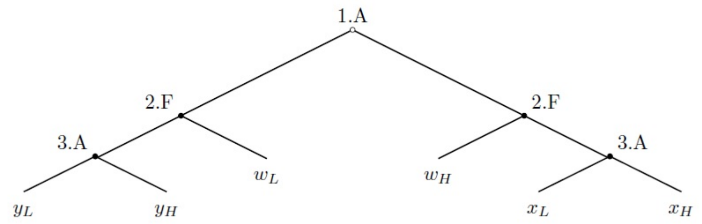


Figure 2.3. Complex three stage mechanism

To show that this SCF can be virtually implemented using restricted perturbations, consider the mechanism represented in Figure 3. This mechanism virtually implements the SCF described above both under complete information and with restricted perturbations. The extra off equilibrium outcomes y_L and y_H ensure that the bad sequential equilibrium that arises in the three stage example described in the previous section does not arise here. Note that in complete information this mechanism implements the allocation w_H if the worker is type L and w_L if the worker is type H.

When restricted information perturbations are introduced, the mechanism fails to implement this SCF exactly. To see this, consider the following equilibrium. Define m_L to be the proportion of low types and m_H to be the proportion of high types. Suppose perturbations happen with probability at most ϵ and that ϵ is sufficiently small. Finally choose mixing probabilities α and β such that the following equations are satisfied:

$$u_P(w_H) = (1-\alpha)m_H\nu(s^H|\theta_H)u_P(x_H) + m_L\nu(s^H|\theta_L)u_P(x_L)$$

$$u_H(w_L) = \Big[\nu(s^L|\theta_H) + \beta\nu(s^H|\theta_H)\Big]u_H(w_H) + (1-\beta)\nu(s^H|\theta^H)u_H(x_H)$$

In the first stage all low types choose the right branch. Meanwhile high types mix, with a proportion α choosing the left branch and a proportion $(1-\alpha)$ choosing the right branch. In the second stage if the worker chose the left branch the firm always chooses w_L. Meanwhile if the worker chose the right branch and the firm observes a signal s^L the firm always chooses w_H. Finally if the worker chose the right branch and the firm observes a signal s^H the firm mixes: with probability β the firm chooses w_H while with probability $(1-\beta)$ the firm proceeds to the third stage. In the third stage a high type worker chooses x_H or y_H while a low type worker chooses x_L or y_L.

It can be easily checked that the strategy profile above outlines a SPE whenever $\epsilon > 0$. In the appendix it is proved that this is indeed the unique SPE. Note that in the first round high types mix between choosing the left branch and the right branch, and so this mechanism does not exactly implement the desired SCF under restricted perturbations. However as $\epsilon \to 0$, then $\alpha \to 1$ where α denotes the fraction of high types who choose the left branch in the first round. This - together with the fact that the SPE outlined above is unique - shows that this mechanism does virtually implement the desired SCF under restricted perturbations. In particular if perturbations are sufficiently small, then the proportion of high types imitating low types can be made to be arbitrarily small. Hence the desired allocation is reached in almost all cases.

This example shows that when exact implementation is prevented by the behaviour of a small proportion of types, allowing players to mix with small probabilities, virtual implementation (as defined above) may still be possible. Note that as the precision of the signal increases, the proportion of players deviating from the complete information equilibrium becomes small. On the one hand - as shown in the previous section - exact and virtual implementation under restricted perturbations coincide when implementation is prevented by the creation of fully pooling 'bad' sequential equilibria. On the other hand, there exist other cases - particularly when perturbations only slightly change equilibrium outcomes - where virtual implementation is more permissive than exact implementation.

2.6 Discussion

The central message of this paper is that the power of SPE implementation depends on the relevant set of information perturbations and the strength of implementation required. At one extreme, if information perturbations are irrelevant and

Table 2.3. Summary (Example 3 can be found in the appendix)

	Exact Implementation	Virtual Implementation
Full Perturbations	Maskin Monotonic	Maskin Monotonic
Restricted Perturbations	Two-stage mechanisms and Example 3	Also Example 2
Complete Information	Condition C	Condition C

there is complete information, a wide range of SCFs can be implemented using Moore-Repullo mechanisms. Meanwhile, at the other extreme, if full perturbations are relevant, then AFHKT show that only Maskin-monotonic SCFs can be implemented. In this paper we have considered the intermediate case of restricted perturbations and provide results which lie somewhere between these two extremes. These results are summarised in Table 3.

The exact power of implementation under restricted perturbations depends on whether virtual implementation or exact implementation is required. One argument for considering virtual implementation is that the definition of exact implementation already allows for mistakes in the rare case when players receive the wrong signal. Hence the concept of exact implementation given here is already - in some sense - a restricted type of virtual implementation, and so it seems natural to instead consider the full version of virtual implementation instead. Meanwhile, an argument for considering exact implementation is that it requires players to follow pure strategies, which are more intuitive than the *almost* pure strategies players follow when considering virtual implementation.

There are two ways in which the results presented here could be easily extended. First the sufficiency result stated here can be extended to an n-player framework where each player moves exactly once. One extra restriction would be necessary: players who move earlier must not be able to communicate information about the preferences of any player who moves later. The proof would be very similar to the two-player case, albeit with extra notation.

The second extension involves considering a class of perturbations wider than those considered in this paper, but still more restricted than than full information restrictions. Note that the formation of 'bad' sequential equilibria relies on players changing their beliefs about their own type to a significant extent. Therefore the results above are also robust to a more general class of restricted perturbations. In particular consider the case where the second-mover receive a signal about their own preferences which is highly (but not perfectly) reliable, while the first-mover receives a significantly less reliable signal. In these cases the second-mover is much more informed than the first-mover, and hence only updates his beliefs about his own preferences by a small amount. This ensures 'bad' sequential equilibria cannot be formed, and that two-stage implementations continue to be robust.

2.7 Appendix

2.7.1 Proof of Proposition 2.4.2

Before proving this theorem we introduce some additional notation and definitions. We use $h_B(s^B, \sigma_A) \in \Sigma_B$ to denote the strategy chosen by player B when he observes signal s^B and player A has chosen strategy σ_A. Hence, $h_B \in H_B$ is a strategy profile of player B, where H_B is the set of all such profiles.

Meanwhile $h_A(s^A, h_B) \in \Sigma_A$ denotes the strategy chosen by player A when he observes signal s^A and expects player B to play according to strategy profile h_B. Hence $h_A \in H_A$ denotes a strategy profile determining the choice of player A when he observes a certain signal and has a certain belief about the strategy profile of player B. H_A is the set of all such strategy profiles. We now define H_B^* and $H_A^*(h_B)$, which denote the possible SPE strategy profiles that occur in a complete information setting:

Definition 16. $h_B \in H_B^*$ *if and only if for all* σ_A*, for all* θ *and for all* $\hat{\sigma}_B \in \Sigma_B$

$$u_B(\Gamma(\sigma_A, h_B(\theta, \sigma_A)); \theta_B) \geq u_B(\Gamma(\sigma_A, \hat{\sigma}_B); \theta_B)$$

Definition 17. $h_A \in H_A^*(h_B)$ *if and only if for all* θ *and for all* $\hat{\sigma}_A \in \Sigma_A$

$$u_A(\Gamma(\sigma_A, h_B(\theta, \sigma_A)), \theta_A) \geq u_A(\Gamma(\hat{\sigma}_A, h_B(\theta, \hat{\sigma}_A)), \theta_A)$$

In a complete information setting with the complete information prior ν, the following proposition is immediately implied by the definitions above:

Proposition 2.7.1. (h_A, h_B) *denote a SPE of* Γ *iff* $h_B \in H_B^*$ *and* $h_A \in H_A^*(h_B)$

This characterizes the SPEs under full information in two stage sequential move games. Note that any sequential move game with finite strategy sets has at least one equilibrium. Hence in order to prove 2.4.2 it is sufficient to show that the strategy profile used in any sequential equilibrium with sufficiently small restricted information perturbations coincides with a SPE in complete information.

To do this consider a game with restricted information perturbations (Γ, ν^ϵ) and corresponding sequential equilibrium strategy profiles $(h^\epsilon_A, h^\epsilon_B)$. It is sufficient to prove that for some $\bar{\epsilon} > 0$, $h^\epsilon_B \in H^*_B$ and $h^\epsilon_A \in H^*_A(h_B)$ whenever $\epsilon \leq \bar{\epsilon}$. The proof is now split into two parts.

First we prove that $h^\epsilon_B \in H^*_B$. This follows from the fact that player B knows his own preferences with certainty and hence in response to player A's move chooses his preferred alternative. [8]

Secondly we prove $h^\epsilon_A \in H^*_A(h^\epsilon_B)$. The proof relies on the fact that player A knows his own type with certainty and estimates the type of player B correctly with probability $(1-\epsilon)$. Hence as $\epsilon \to 0$ the incentives of player A are very similar to the incentives he has in complete information. In particular the probability ϵ event where he estimates the type of player B incorrectly becomes relatively unimportant.

We slightly abuse notation by defining $u_A(\sigma_A, \sigma_B; \theta_A) := u_A(\Gamma(\sigma_A, \sigma_B); \theta_A)$. Moreover throughout the proof we use the fact that perturbations are restricted: that is to say $s^A_A = \theta_A$ and $s^B_B = \theta_A$.

Proof of 2.4.2 Part (i): $h^\epsilon_B \in H^*_B$

Proof. Suppose this does not hold. Then for some signal $\tilde{s}^B$, and strategy $\tilde{\sigma}_A$ there exists a deviating strategy $\hat{\sigma}_B$ such that:

$$u_B(\tilde{\sigma}_A, h^\epsilon_B(\tilde{s}^B, \tilde{\sigma}_A); \tilde{s}^B_B) < u_B(\tilde{\sigma}_A, \hat{\sigma}_B; \tilde{s}^B_B)$$

Since information perturbations are restricted $s^B_B = \theta_B$, and it follows that:

$$u_B(\tilde{\sigma}_A, h^\epsilon_B(\tilde{s}^B, \tilde{\sigma}_A); \theta_B) < u_B(\tilde{\sigma}_A, \hat{\sigma}_B; \theta_B)$$

[8]Note that this is the part of the proof that does not hold in the setting AFHKT consider. In their setting player B may infer something about his own preferences from the move of player A. In particular, $u_B(\Gamma(\sigma_A, \sigma_B); \theta_B) \neq u_B(\Gamma(\sigma_A, \sigma_B); s^B_B)$.

Consider the following strategy profile:

$$\hat{h}^{\epsilon}_B(s^B, \sigma_A) = \begin{cases} \hat{\sigma}_B & \text{if} \quad (s^B, \sigma_A) = (\tilde{s}^B, \tilde{\sigma}_A) \\ h^{\epsilon}_B(s^B, \sigma_A) & \text{otherwise} \end{cases}$$

Playing according to strategy profile $\hat{h}^{\epsilon}_B$ rather than strategy profile h^{ϵ}_B leads to a higher payoff in the subgame when $(s^B, \sigma_A) = (\tilde{s}^B, \tilde{\sigma}_A)$ and the same payoff otherwise. Hence h^{ϵ}_B cannot be a sequential equilibrium profile of the game (Γ, ν^{ϵ}). This is a contradiction, and completes the proof.

□

Proof of 2.4.2 Part (ii): $h^{\epsilon}_A \in H^*_A(h^{\epsilon}_B)$

Proof. First define the following:

$$\begin{aligned}
\underline{u} &:= \min_{x, s^A}\{u_A(x; s^A_A)\} \\
\overline{u} &:= \max_{x, s^A}\{u_A(x; s^A_A)\} \\
\sigma_A(s^A) &= h^{\epsilon}_A(s^A, h^{\epsilon}_B) \\
u(s^A) &:= u_A(\sigma_A(s^A), h^{\epsilon}_B(s^A, \sigma_A(s^A)); s^A_A) \\
\hat{u}(s^A) &:= \max_{\tilde{\sigma}_A}\{u_A(\tilde{\sigma}_A, h^{\epsilon}_B(s^A, \tilde{\sigma}_A); s^A_A)\}
\end{aligned}$$

We use $\overline{u}$ and $\underline{u}$ to refer to the maximum and minimum payoffs player A could receive, while $u(s^A)$ is the utility player A obtains when he plays according to strategy $\sigma_A(s^A) = h_A(s^A, h^{\epsilon}_B)$, player B has the same signal as him ($s^B = s^A$) and plays according to a strategy profile h^{ϵ}_B. Meanwhile $\hat{u}(s^A)$ is the maximum utility player A could obtain in this situation by choosing some arbitrary strategy. Let $\hat{\sigma}_A(s^A)$ be one of these maximizing strategies, so that $\hat{u}(\theta) = u_A(\hat{\sigma}_A(\theta), h^{\epsilon}_B(\theta, \hat{\sigma}_A(\theta)); \theta)$.

Now suppose $h^{\epsilon}_A \notin H_A(h^{\epsilon}_B)$. Remembering that h_A is a strategy profile of a sequential equilibrium, we aim for a contradiction. Since $h^{\epsilon}_A \notin H_A(h^{\epsilon}_B)$, it follows that for some signal $\tilde{s}^A$ there exists a profitable deviation $\tilde{\sigma}_A$. That is to say:

$$u_A(\sigma_A(\tilde{s}^A), h_B^\epsilon(\tilde{s}^A, \sigma_A(\tilde{s}^A)); \tilde{s}_A^A) < u_A(\tilde{\sigma}_A, h_B^\epsilon(\tilde{s}^A, \tilde{\sigma}_A); \tilde{s}_A^A) \tag{2.1}$$

Using the definition of $\hat{\sigma}_A$, note that the strategy $\hat{\sigma}_A(s^A)$ maximizes the payoff of player A given his signal is s^A. Therefore:

$$u_A(\tilde{\sigma}_A, h_B^\epsilon(\tilde{s}^A, \tilde{\sigma}_A); \tilde{s}_A^A) \leq u_A(\hat{\sigma}_A(\tilde{s}^A), h_B^\epsilon(\tilde{s}^A, \hat{\sigma}_A(\tilde{s}^A)); \tilde{s}_A^A) \tag{2.2}$$

Putting these equations 2.1 and 2.2 together and using the definition of $u(s^A)$ and $\hat{u}(s^A)$ leads to the following:

$$\begin{aligned} u_A(\sigma_A(\tilde{s}^A), h_B^\epsilon(\tilde{s}^A, \sigma_A(\tilde{s}^A)); \tilde{s}_A^A) &< u_A(\hat{\sigma}_A(\tilde{s}^A), h_B^\epsilon(\tilde{s}^A, \hat{\sigma}_A(\tilde{s}^A)); \tilde{s}_A^A) \\ u(\tilde{s}^A) &< \hat{u}(\tilde{s}^A) \end{aligned}$$

Now let $\delta = \hat{u}(\tilde{s}^A) - u(\tilde{s}^A)$ and note that $\delta > 0$. Define an alternative strategy profile $\hat{h}_A^\epsilon$ as follows:

$$\hat{h}_A^\epsilon(s^A) = \begin{cases} \hat{\sigma}_A(s^A) & \text{when} \quad s^A = \tilde{s}^A \\ \sigma_A(s^A) & \text{when} \quad s^A \neq \tilde{s}^A \end{cases}$$

We now show that $\hat{h}_A^\epsilon$ is a profitable deviation. When $s^A \neq \tilde{s}^A$, payoffs under both strategy profiles are equal under both strategy profiles z, so we focus on the case where $s^A = \tilde{s}^A$. Note that in this case $\hat{h}_A^\epsilon(s^A) = \hat{\sigma}_A(\tilde{s}^A)$ and $h_A^\epsilon(s^A, h_B^\epsilon) = \sigma_A(\tilde{s}^A)$. Since information perturbations are restricted, $\theta_A = \tilde{s}_A^A$. Hence it is enough to show that:

$$S = E_{s^B \in \Theta_B}[u_A(\hat{\sigma}_A(\tilde{s}^A), h_B^\epsilon(s^B, \hat{\sigma}_A(\tilde{s}^A)); \tilde{s}_A^A)] - E_{s^B \in \Theta_B}[u_A(\sigma_A(\tilde{s}^A), h_B^\epsilon(s^B, \sigma_A(\tilde{s}^A)); \tilde{s}_A^A)] > 0$$

First note that with probability $p > (1-\epsilon)$, $s^B = \tilde{s}^A$. In this case the left hand side is equal to $\hat{u}(\tilde{s}^A)$, while the right-hand side is equal to $u(\tilde{s}^A)$. Moreover with probability ϵ any payoff between $u_A \in [\underline{u}, \overline{u}]$ may be obtained. These observations lead to the following bounds:

$$\begin{aligned} E_{s^B \in \Theta_B}[u_A(\hat{\sigma}_A(\tilde{s}^A), h_B^\epsilon(s^B, \hat{\sigma}_A(\tilde{s}^A)); \tilde{s}_A^A)] &\geq (1-\epsilon)\hat{u}(\tilde{s}^A) + \epsilon \underline{u} \\ E_{s^B \in \Theta_B}[u_A(\sigma_A(\tilde{s}^A), h_B^\epsilon(s^B, \sigma_A(\tilde{s}^A)); \tilde{s}_A^A)] &\leq (1-\epsilon)u(\tilde{s}^A) + \epsilon \overline{u} \end{aligned}$$

Using these bounds, the fact that $\delta = \hat{u}(\tilde{s}^A) - u(\tilde{s}^A) > 0$ and assuming $\epsilon < \frac{1}{2}$ gives:

$$\begin{aligned} S &\geq (1-\epsilon)\hat{u}(\tilde{s}^A) + \epsilon\underline{u} - (1-\epsilon)u(\tilde{s}^A) - \epsilon\overline{u} \\ &> \delta - 2\epsilon(\overline{u} - \underline{u}) \end{aligned}$$

$\delta > 0$ and both δ and $(\overline{u} - \underline{u})$ are fixed parameters. Therefore there exists some $\overline{\epsilon}$ such that $S > 0$ whenever $\epsilon \in (0, \overline{\epsilon})$. This shows that $\hat{h}_A^\epsilon$ is a profitable deviation and hence h_A^ϵ cannot be the strategy profile of a sequential equilibrium. This proves the result. □

2.7.2 Example: $F2$ is sufficient but not necessary

Consider again the initial example of the firm and the worker. Now however there is a third type of worker, ($\theta_B = S$). This worker has an outside option that he prefers to w_H, but otherwise has the same preferences as the high type worker. This outside option can be thought of as another job offer with a high salary. In case he does not reach an agreement with the firm he takes the outside offer. Also suppose that there are two types of firms ($\theta_A \in \{Y, N\}$). One firm would like to hire this special worker by offering him a wage that is even higher than the outside option. The other type of the firm does not want to pay such a high wage.

The references are given in Table 4.

The social choice function where $f(N, L) = f(Y, L) = w_L$, $f(N, H) = f(Y, H) = w_H$, $f(N, S) = d$ and $f(Y, S) = S$ can be implemented in three stages in complete information, where the worker first chooses between the special branch and the normal branch. In case the worker has chosen the special branch, the firm decides

Table 2.4. Example: F2 is not necessary

	Preferences	
Norm fi	$u_P(w_L;\theta) > u_P(w_H;\theta) > u_P(d;\theta) > u_P(S;\theta)$	for $\theta \in \{(N,L),(N,H),(Y,L),(Y,H)\}$
Spec fi	$u_P(w_L;\theta) > u_P(w_H;\theta) > u_P(S;\theta) > u_P(d;\theta)$	for $\theta \in \{Y,S\}$
	$u_P(w_L;\theta) > u_P(w_H;\theta) > u_P(S;\theta) > u_P(d;\theta)$	for $\theta \in \{(N,L),(N,H)\}$
Low t	$u_A(S,\theta) > u_A(w_H;\theta) > u_A(w_L;\theta) > u_A(d;\theta)$	for $\theta_B = L$
High t	$u_A(S,\theta) > u_A(w_H;\theta) > u_A(d;\theta) > u_A(w_L;\theta)$	for $\theta_B = H$
Spec t	$u_A(S,\theta) > u_A(w_H;\theta) > u_A(d;\theta) > u_A(w_L;\theta)$	for $\theta_B = S$

to pay a very high wage S if the worker is indeed the special type and the firm is special, too. It chooses outside option d otherwise. On the other hand, if the worker chooses the normal branch, the game continues as in the basic example.

If the proportion of special firms is sufficiently small and normal workers dislike allocation O sufficiently, then this mechanism is robust to restricted perturbations and the SCF can be implemented robustly, despite requiring three stages. Note however, that this mechanism can be reduced to two stages, when allowing players to move simultaneously in the first stage. Workers report that they are *normal* or *special* and the firm chooses one of S and the default d and one of w_H and w_L. If the worker chooses the special branch the game ends and S or d as chosen by the firm is implemented. If the worker chooses the normal branch then if the firm chose w_H this is implemented. In the final case, where the worker has chosen the normal branch and the firm chose w_L, the worker gets to make a final choice between accepting w_L and rejecting the offer to implement the default d.

2.7.3 Simultaneous moves

We now provide an example to show that the credible threat condition is not necessary for robust implementation under restricted information perturbations when allowing players to move simultaneously.

Consider the case where there are two players A and B. For simplicity assume that the preferences of player B are fixed, while player A's preferences are given by θ or $\hat{\theta}$. We assume that player A knows his preferences with certainty while the signal player B receives is equal to player A's preferences with probability $1 - \epsilon$ and equal to the other preference with the remaining probability ϵ. Now consider the following mechanism:

		B: θ	B: $\hat{\theta}$
A	θ	Γ	$(0, 0, 0)$
A	$\hat{\theta}$	$\theta : (1, 0, 10)$ $\hat{\theta} : (0, 1, 0)$	$(7, 7, 3)$

Figure 2.4. Simultaneous moves

In the first stage of the game both players simultaneously choose between reporting θ and reporting $\hat{\theta}$. This is described in Figure 2.4. If both players report $\hat{\theta}$ then the game ends and players receive the payoffs given in brackets. The first number corresponds to the payoff of player A if he is type θ, the second number is the payoff of player A if he is type $\hat{\theta}$ and the third number is the payoff of player B. Similarly if player B reports $\hat{\theta}$ and player A reports θ, the payoffs are $(0, 0, 0)$ and the game ends.

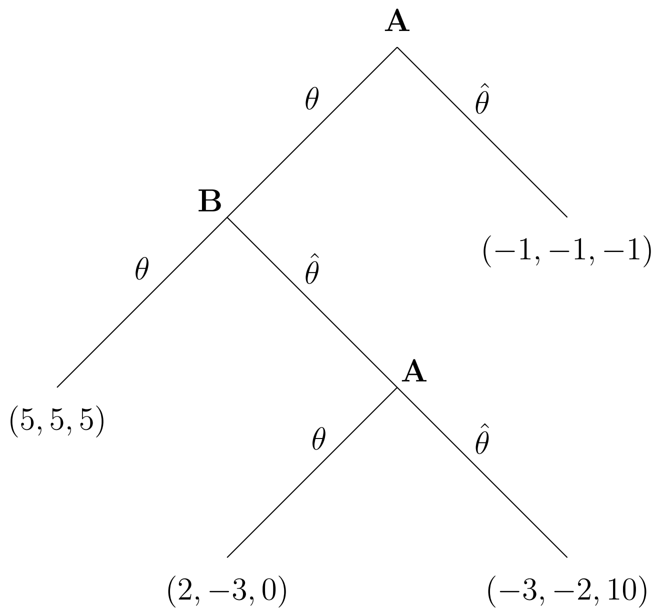


Figure 2.5. Mechanism Γ

Now consider the case where player A reports $\hat{\theta}$ and player B reports θ. In this case player A has got a second move and chooses again between the reports θ and $\hat{\theta}$ which correspond to payoff vectors of $(1,0,10)$ and $(0,1,0)$ respectively.

In the case where both players report θ, they start playing the mechanism Γ given by the game tree in Figure 2.5 in the second stage.

It can easily be checked that the underlying preferences do not satisfy the credible threat condition.

We now show that despite this fact, the simultaneous move mechanism described above robustly implements the social choice function with payoffs $(5,5)$ in state θ and $(7,3)$ in state $\hat{\theta}$ under restricted information perturbations. For simplicity we assume that the states θ and $\hat{\theta}$ are ex-ante equally likely.

First note that the unique equilibria under complete information are given by the reports $(\theta,\theta,\theta,\theta)$ in state θ and $(\hat{\theta},\hat{\theta})$ in state $\hat{\theta}$. Hence the desired SCF is implemented under complete information.

Now consider the case where player A's realised preferences are θ. If the mechanism

Γ is reached, player A has got a dominant strategy to re-report his preferences as θ. Moreover whenever player A's preferences are θ his initial report is θ. This ensures him a payoff of 2 which is greater than any payoff he can hope to achieve by reporting $\hat{\theta}$, since the reports $(\hat{\theta}, \hat{\theta})$ are not an equilibrium. Knowing this, player B assigns a high probability to player A's preferences being θ whenever he observes A re-reporting himself as θ and mechanism Γ is played. As a consequence B also reports θ and the desired allocation is implemented. There cannot be a case, where player A re-reports his preferences as θ and player B then assigns a higher probability to A's preferences being $\hat{\theta}$ than before the first stage.

Secondly consider the case where player A's preferences are given by $\hat{\theta}$. Then the reports $(\hat{\theta}, \hat{\theta})$ are an equilibrium. Player A cannot gain by deviating as there does not exist an allocation which gives him a higher payoff. Player B cannot gain by deviating to another report either: If he reports θ player A has got another move where he has a dominant strategy to re-report $\hat{\theta}$, leaving player B with a payoff $0 < 3$. Hence the report $(\hat{\theta}, \hat{\theta})$ is an equilibrium if the state is $\hat{\theta}$.

Note also that it is the only equilibrium in this state. In particular the mechanism Γ played when the reports are (θ, θ) cannot be an equilibrium, as it would implement an allocation $(-1, -1)$, which neither of the players likes.

2.7.4 Virtual Implementation

We now prove that the SPE equilibrium in mixed strategies stated in section 5.1 is indeed the unique equilibrium of the mechanism described and hence virtually implements the desired SCF.

Proof. Let $\delta = \sqrt{\epsilon}$ and suppose ϵ is sufficiently small. In this case, if more than fraction δ of low types choose the left branch, the principal - on observing signal s^L will challenge the report. This is because the report is sufficiently likely to originate from a low type and hence:

$$u_P(w_L) < \frac{\delta(1-\epsilon)m_L}{\delta(1-\epsilon)m_L + \epsilon m_H} u_P(y_L) + \frac{\epsilon m_H}{\delta(1-\epsilon)m_L + \epsilon m_H} u_P(y_H)$$

Secondly note that if more than fraction δ of low types choose the right branch, the principal - on observing signal s^L will accept the report. This is because the report is sufficiently likely to originate from a low type and hence:

$$u_P(w_H) > \frac{\delta(1-\epsilon)m_L}{\delta(1-\epsilon)m_L + \epsilon m_H} u_P(x_L) + \frac{\epsilon m_H}{\delta(1-\epsilon)m_L + \epsilon m_H} u_P(x_H)$$

Suppose there is a SPE where more than δ low types choose the left branch in the first round. Then these low types with probability greater than $(1-\epsilon)$ would receive payoff $u_L(y_L)$. If ϵ is sufficiently low, it is optimal for these low types to deviate and choose the right branch in the first round guaranteeing a payoff higher than $u_L(y_L)$. It follows that in any SPE a fraction at least $(1-\delta)$ low types chooses the right branch in the first stage.

Since a high fraction of low types report L in the first round, it follows from above that the principal - on observing a report of L and signal s^L - will always accept the report and implement w_H. Since this is the highest payoff a low type can receive it follows that all low types will report L in the first round.

Since only high types choose the left branch, it follows that the firm will accept to pay w_L, whenever a worker chooses the left branch in the first stage. Therefore high type workers have a choice between (i) choosing the left branch and receiving a guaranteed payoff of $u_H(w_L)$ and (ii) choosing the right branch. Suppose all high types choose the right branch. Then the firm - on observing a worker has chosen the right branch and a signal s^H - will challenge the worker by moving to the third stage - and x_H will be implemented. In this case high types - preferring w_H to x_H - would have an incentive to deviate and choose the left branch initially. Suppose now on the other hand that all high types choose the left branch. Then the firm - on observing that the right branch has been chosen and a signal s^H - will not challenge and w_H will be implemented. In this case high types - preferring w_H to w_L - would have an incentive to deviate.

It follows from the two observations above that high types must mix in the first stage. Moreover for high types to be indifferent over their mixing, it follows that the principal must mix in the second stage after observing a report L and a signal s^H. The mixing parameters α and β are calculated above, and hence this is the unique SPE.

$\square$

Chapter 3

Generalised Weighted Raiffa Solutions

3.1 Introduction

In the 1950s Nash (1950) and Raiffa (1953) independently introduced two solution concepts to a general bargaining problem, which has since been studied extensively. While Nash motivated his solution by appealing to the independence of irrelevant alternatives (IIA) axiom, Raiffa focused on the interim steps necessary to reach a final agreement. Both of these original solutions require that identical players are treated equally, and effectively assume that all players have equal bargaining weight. However in many situations this is not the case, and one party can influence the outcome of negotiations more effectively than another due to different levels of skill or commitment.

Acknowledging this fact, Harsanyi and Selten (1972) introduced a family of weighted Nash solutions, where each player is assigned a bargaining weight. These bargaining weights concisely model differences between players, which are not related to the players' payoff functions. Firstly one can think of the bargaining weights reflecting player specific characteristics: a player who is more patient or who can quickly respond with a counter-proposal might be expected to achieve a higher allocation than a player who is less patient or takes longer to respond to proposals. Secondly differences in bargaining weight may reflect differences in the negotiation procedure or institutional framework. For instance permanent members of the UN security council have a veto while other countries do not, and this system means that permanent members are in a stronger position to negotiate favourable agreements than non-permanent members. This paper extends the original solution proposed by Raiffa (1953) to accommodate the case where players have asymmetric bargaining weights.

Weighted versions of other bargaining solutions such as the egalitarian solution (Thomson (1994)) and Kalai-Smorodinksy solution (Kalai (1977), Thomson (1994), Dubra (2001)) have also been proposed. Although the Raiffa solutions have received considerable attention (see for example Anbarci and Sun (2013) and Trockel (2015)) - as far as we are aware - these solutions have not been generalised to accommodate unequal bargaining weights. In this paper we introduce and provide cooperative and non-cooperative foundations for a family of weighted Raiffa solutions. The cooperative foundation appeals to two of the original axioms proposed by Nash and a monotonicity axiom focusing on interim agreements. Meanwhile the non-cooperative foundation shows that these solutions can be implemented using simple bargaining models where offers are made either intermittently or where the identity of the proposer is persistent.

Weighted bargaining solutions are used in many economic applications. Prominent examples include wage bargaining in labour economics (see for example Shimer (2005))) and bankruptcy negotiations in the finance literature (see Yue (2010)). The vast majority of these applications use the weighted Nash solution, due to its strong cooperative and non-cooperative foundations. We show that weighted Raiffa solutions have similarly strong foundations and hence should be considered as an alternative. In particular considering these different solutions alongside each other could serve as a robustness check, and help determine whether the predictions of a model are sensitive to the solution concept used. Furthermore our results underline the fact that bargaining models with patient players and relatively close deadlines are associated with the Raiffa solution, while bargaining models with relatively impatient players and distant deadlines are associated with the Nash solution. This has implications for the design of negotiation protocols, since policy makers may be able to affect the outcome of negotiations simply by changing the timing of the deadline.

The second section outlines a characterisation for the family of weighted Raiffa solutions. It is related to the characterisation provided by Diskin et al. (2011) who introduce and characterise a family of p-Raiffa solutions. This family includes the discrete and continuous Raiffa solutions introduced by Raiffa (1953) as the extreme cases when $p = 1$ and as p approaches 0 respectively. We build on this approach with two important differences. Firstly, we do not use a symmetry axiom in order to characterise a family of weighted (λ, p)-Raiffa solutions. Secondly, rather than axiomatizing on the sequence of interim agreements, our characterisation uses weaker axioms related to the eventual bargaining solution wherever possible. These weaker axioms are commonly used in bargaining theory, making it easier to compare with existing results.

In order to be able to make further comparisons we also provide a new axiomatization for weighted Kalai-Smorodinsky solutions which uses similar axioms. This helps to show how the *weighted Raiffa solution*, the *weighted Kalai-Smorodinsky solution* and the *weighted egalitarian solution* can all be axiomatized by appealing to different versions of a monotonicity axiom combined with scale invariance and Pareto optimality.

The third section provides non-cooperative foundations for weighted Raiffa solutions. We introduce a class of bargaining models that can be used to approximately implement any weighted Raiffa solution. The games considered have a finite number of rounds, where players do not discount and the identity of the proposer is determined by a Markov process with $(n + 1)$ states. In state i player i makes a proposal, while in state $(n+1)$ no offer is made. If an offer is accepted by all other players then it is implemented. Otherwise negotiations continue to the next round. In particular we show that specific types of this general model - when offers are intermittent or the identity of the proposer is persistent - implement a weighted Raiffa solution.

This class of bargaining models generalises the finite horizon models studied by Stahl (1972) and Sjostrom (1991). Moreover it is also related to the infinite horizon model considered by Binmore et al. (1986) which implements the Nash solution. In particular our model has a strong resemblance with the generalised version of this model considered by Britz et al. (2010). While they consider an infinite horizon model with a discount factor where the identity of the proposer is determined by a Markov process, we consider a similar environment with a finite horizon and no discount factor. This shows that when discounting is the dominant factor a weighted Nash solution is implemented, while when deadlines are the dominant factor a weighted Raiffa solution is implemented. Hence our analysis extends the results in Gomes et al. (1999) and Imai and Salonen (2012), who study this effect in settings where all players are equally likely to be selected as proposer in each round. We now outline the bargaining problem and introduce the family of Raiffa solutions.

3.1.1 The bargaining problem

Consider n-player bargaining problems where the set of players is denoted by $N = \{1, , \ldots, n\}$. Players negotiate over how to split a cake of size one with free disposal. The default allocation is normalised to $\mathbf{0} \in \mathrm{Re}^n$, while the set of feasible allocations $X \subset \mathrm{Re}^n$ is defined as follows:

$$X = \left\{ \mathbf{x} : \sum x_i \leq 1 \text{ and } x_j \geq 0 \text{ for all } j \in N \right\}$$

Each player i has a utility function $u_i : [0, 1] \mapsto \mathrm{Re}$, which maps a quantity x_i to a payoff s_i. We assume that all u_i are strictly increasing and strictly concave.[1] Using these utility functions, the set of feasible payoff vectors $S \subset \mathrm{Re}^n$ is defined as follows:

$$S = \left\{ \mathbf{s} : \text{ there exists } \mathbf{x} \in X \text{ such that } s_i = u_i(x_i) \text{ for all } i \in N \right\}$$

Since the utility functions are concave and X is a simplex, it follows that S is convex. Define the default allocation of player i to be $d_i = u_i(0)$ and let $\mathbf{d} = (d_1, d_2, ..., d_n)$ be the vector of all players' default allocations. We refer to the pair

[1]This models situations where all players strictly prefer more cake to less cake and are strictly risk averse.

$(S, \mathbf{d})$ as the utility representation of a bargaining problem. An n-player bargaining solution Φ maps a utility representation $(S, \mathbf{d})$ to a solution $\Phi(S, \mathbf{d}) \in S$. This final requirement captures the fact that a bargaining solution Φ selects a feasible payoff vector. When $\mathbf{s} \in S$, we define the ideal point for player i as $m_i(\mathbf{s}|S)$. This is the highest utility level that player i could receive while still ensuring that each player j receives a utility of at least s_j.

$$m_i(\mathbf{s}|S) = \max\{\hat{s}_i | \hat{\mathbf{s}} \in S \text{ and } \hat{s}_j \geq s_j \text{ for all } j \in N\}$$

We define $m(\mathbf{s}|S)$ to be the vector of such utility levels, and refer to this as the *ideal point given* $\boldsymbol{s}$. Note that by convexity of S, $m_i(\mathbf{s}|S)$ is strictly decreasing in s_j whenever $j \neq i$. This captures the fact that players are in a competitive situation: if the utility allocation s_j of an opponent $j \neq i$ increases, keeping that of the remaining players at least constant, then the highest feasible utility of player i - namely $m_i(\mathbf{s}|S)$ is reduced.

3.1.2 The family of Raiffa solutions:

Ideal points have been used by Raiffa (1953) and Diskin et al. (2011) to define bargaining solutions as a limit of an iterative process, and below we extend this approach. The initial point of the iteration is fixed to be the default allocation $m_i(\mathbf{s}(k)|S)$. Subsequent steps $\mathbf{s}(k+1)$ are determined by the current disagreement point $\mathbf{s}(k)$ and the current ideal point $m(\mathbf{s}(k)|S)$. This captures the fact that during a negotiation some player i considers two pieces of information: first he focuses on what he is sure to obtain, namely the current disagreement point $\mathbf{s}_i(k)$; secondly he focuses on what he could possibly obtain in an ideal world namely $\mathbf{s}_i(k)$. Finally the bargaining solution $\Phi(S, \mathbf{d})$ is given to be the limit of this process. Each step in the iterative process can be thought of as an interim agreement, where every player prefers the interim agreement $\mathbf{s}(k+1)$ to the interim agreement $\mathbf{s}(k)$.

The two-player discrete Raiffa solution was the first iterative bargaining solution to be introduced in Raiffa (1953). Here the new agreement $\mathbf{s}(k+1)$ is calculated by taking the midpoint between the current agreement $s(k)$ and the current ideal point $m(S, s(k))$. By convexity of S this point lies in S.

Definition 18 (Discrete Raiffa solution)**.**
The two-player discrete Raiffa solution is defined as $\Phi(S, \boldsymbol{d}) = \lim_{k\to\infty} \boldsymbol{s}(k)$ *where:*

- $\boldsymbol{s}(0) = \boldsymbol{d}$
- $\boldsymbol{s}(k+1) = \frac{1}{2}\boldsymbol{s}(k) + \frac{1}{2}m(\boldsymbol{s}(k), S)$

This solution is illustrated in Figure 3.1.

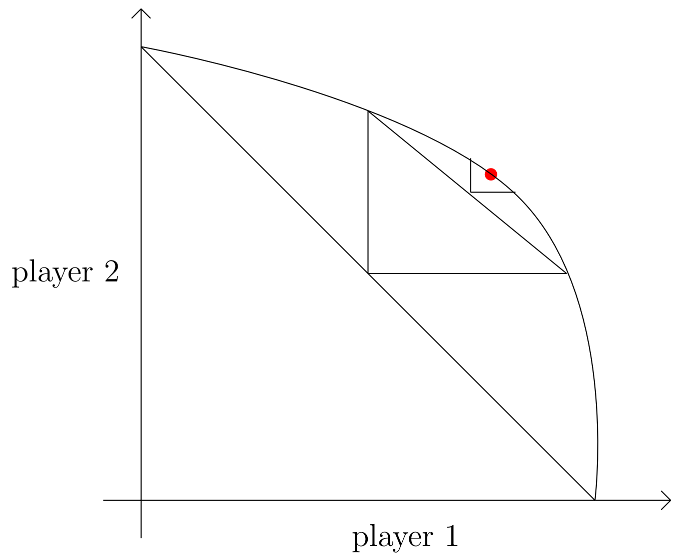


Figure 3.1. Discrete Raiffa solution

Salonen (1988) extends this solution to n-players, and provides an axiomatization for these n-player discrete Raiffa solutions. While in the two-player case each player moves half way towards his current ideal point, for the n-player case this may lead to payoffs which are not feasible. Therefore in the n-player case each player moves a fraction $\frac{1}{n}$ towards their ideal point.

A more general set of n-player Raiffa solutions is considered by Diskin et al. (2011). The Raiffa solutions in this family are characterized by a parameter p determining the step size of interim agreements and hence the speed of convergence to a solution. When $p = 1$, the p-Raiffa solution corresponds to the discrete Raiffa solution. However when $p < 1$ interim agreements lie closer together, and the sequence $\mathbf{s}(k)$ converges more slowly. As $p \to 0$, the interim agreements become arbitrarily close.

This approximates the n-player continuous Raiffa solution. The two player version of this solution was introduced by Raiffa (1953) and axiomatized by Peters and Damme (1991). The family of solutions considered by Diskin et al. (2011) can be stated as follows:

Definition 19 (p-Raiffa solution)**.**
For $p \in (0,1]$ the n-player p-Raiffa solution is defined as $\Phi^p(S, \boldsymbol{d}) = \lim_{k\to\infty} \boldsymbol{s}(k)$ where:

- $\boldsymbol{s}(0) = d$
- $\boldsymbol{s}(k+1) = \left(1 - \frac{p}{n}\right)\boldsymbol{s}(k) + \frac{p}{n} m(S, \boldsymbol{s}(k))$

For the two-player case with bargaining weights equal to 0.7 and 0.3 respectively, this is illustrated in Figure 3.2.

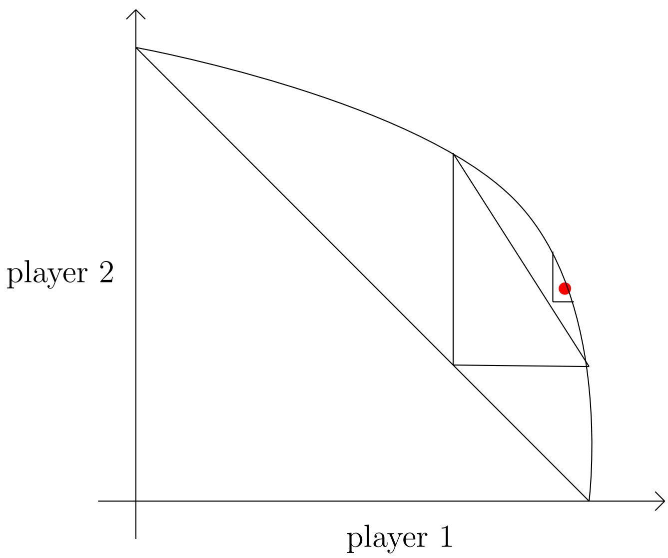


Figure 3.2. P-Weighted Raiffa solution P =(0.7,0.3)

Our contribution is to generalise this set of Raiffa solutions to a more general set of weighted bargaining solutions where players may have different bargaining weights. A bargaining weight $\lambda_i \in (0,1)$ is assigned to each player, and the parameter $\lambda = (\lambda_1, \lambda_2 ..., \lambda_n)$ denotes the vector of players' exogenous bargaining weights. Without loss of generality we assume that $\Sigma\lambda_i = 1$, and define (λ, p)-Raiffa solutions as follows:

Definition 20 ((λ, p)-Raiffa solution)**.**
The n-player (λ, p)-Raiffa solution is defined as $\Phi^{\lambda,p}(S, \boldsymbol{d}) = \lim_{k\to\infty} s(k)$ *where:*

- $\boldsymbol{s}(0) = \boldsymbol{d}$
- $\boldsymbol{s}_i(k+1) = \Big(1 - p\lambda_i\Big)\boldsymbol{s}(k) + p\lambda_i m(\boldsymbol{s}(k)|S)$

For the two player case, where $\lambda = (0.75, 0.25)$ and $p = 0.8$, this solution is illustrated in Figure 3.3.

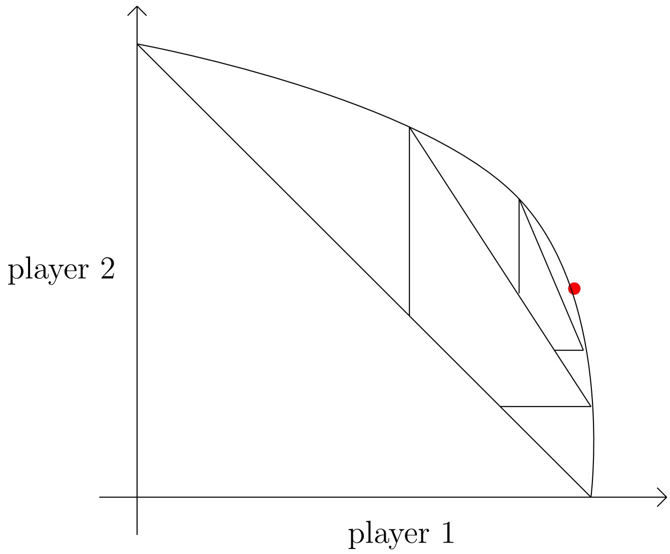


Figure 3.3. Weighted Generalised Raiffa solution: (0.75, 0.25, 0.2)

The next two sections provide a cooperative and a non-cooperative foundation for this family of bargaining solutions.

3.2 Cooperative foundation

In this section we provide a cooperative foundation for (λ, p)-Raiffa solutions. First we introduce some additional notation. The set of individually rational points that all players prefer over the default is referred to as $S_{\mathbf{d}} := \{\mathbf{s} \in S | s_i \geq d_i \text{ for all } i\}$. We use $PF(S) := \{\mathbf{v} \in S | m(\mathbf{v}|S) = \mathbf{v}\}$ to refer to those points in S that are Pareto optimal. It is further assumed that any point which is weakly Pareto optimal is also strictly Pareto optimal - ie $m_i(\mathbf{v}|S) = \mathbf{v}_i$ implies $\mathbf{v} \in PF(S)$.[2]

Unlike other solution concepts, the Raiffa solution explicitly specifies a sequence of interim points between the default allocation and the bargaining solution. These interim points can be thought of as a series of interim agreements that players make before reaching the final solution. To capture this idea that a bargaining solution may be reached through a number of interim agreements, Diskin et al. (2011) model interim solutions by using a step function. In a similar way we say that an interim solution maps a bargaining problem $(S, \mathbf{s})$ to a unique point $\delta(S, \mathbf{s}) \in S$ whenever $\mathbf{s} \in S$. An interim solution $\delta(S, \mathbf{d})$ can be interpreted as a first interim agreement that players reach as they move towards the eventual bargaining solution $\Phi(S, \mathbf{d})$. We say that δ is associated with the bargaining solution Φ if repeated applications of the interim solution δ eventually approximate the bargaining solution Φ:

Definition 21 (Interim solutions δ)**.**
An interim solution δ is associated with a bargaining solution Φ iff $\Phi(S, \boldsymbol{d}) = \lim_{k\to\infty} \boldsymbol{d}^k$ where $\boldsymbol{d}^0 = \boldsymbol{d}$ and $\boldsymbol{d}^{k+1} = \delta(S, \boldsymbol{d}^k)$ for all $(S, \boldsymbol{d})$

Note that the interim solution $\delta = \Phi$ is trivially associated with the bargaining solution Φ. Hence it is clear that any bargaining solution Φ is associated with at least one interim solution. We say an interim solution δ is non-trivial if $\mathbf{d} \notin PF(S)$ implies $\delta(S, \mathbf{d}) \notin PF(S)$. Following most of the literature we focus only on Pareto optimal and scale-invariant bargaining solutions. Furthermore we require that any points which are not individually rational do not affect the final bargaining solution. This leads to the following axioms:

Scale invariance (SI)
If $\mathbf{b}, \mathbf{c} \in \mathrm{Re}^n_+$ and $F(\mathbf{x}) = \mathbf{b} \cdot \mathbf{x} + \mathbf{c}$, then $\Phi(F(S), F(d)) = F(\Phi(S, d))$

Pareto optimality (PO)
If $s'_i > \Phi_i(S, d)$ for all $i \in N$, then $\mathbf{s}' \notin S$

Irrelevance of non-individually rational points (IIR)

$$\Phi(S, \mathbf{d}) = \Phi(T, \mathbf{d}) \quad \text{whenever} \quad S_{\mathbf{d}} = T_{\mathbf{d}}$$

[2] Note that this follows immediately if the bargaining set S is associated with strictly increasing utility functions

Along with these standard axioms, we introduce an axiom requiring that if any feasible allocation $s \in S_{\mathbf{d}}$ remains feasible when the bargaining set changes from S to T, then such a change is weakly beneficial for all players. Therefore if additional allocations become feasible, then no player becomes worse off. This requirement is captured as follows:

Φ-monotonicity (Φ-MON)

$$\text{If} \quad S \subseteq T, \quad \text{then} \quad \Phi_i(S, \mathbf{d}) \leq \Phi_i(T, \mathbf{d})$$

We say that moving from S to T is an *enlargement of the feasible set* if and only if $S \subseteq T$. Hence the Φ monotonicity condition says that if the feasible set is enlarged, then no player becomes worse off. Although this axiom seems plausible, it is not compatible with Pareto optimality and scale invariance. This is shown by the following result due to Thomson (1994):

Proposition 3.2.1 ((λ)-Egalitarian solutions)**.**
If Φ satisfies (IIR), (PO) and (Φ-MON) then Φ is a (λ)-weighted Egalitarian solution. The (λ)-weighted Egalitarian solution is not scale invariant and is defined uniquely as follows:

$$EG_{\lambda}(S, \mathbf{d}) \in \Big\{ \mathbf{s} | s_i = d_i + k\lambda_i \;\; \textit{for some} \;\; k \in \mathrm{Re}_+ \Big\} \cap PF(S)$$

This shows that the axiom of Φ-monotonicity is too strong to characterise scale-invariant and Pareto optimal solutions. In light of this result, we consider the following weaker axiom of δ-monotonicity:

δ-monotonicity (δ-MON) For some interim solution δ associated with Φ:

$$\text{If} \quad S \subseteq T, \quad \text{then} \quad \delta_i(S, \mathbf{d}) \leq \delta_i(T, \mathbf{d}) \text{ for all } i$$

The δ-monotonicity axiom also appeals to the idea that enlarging the feasible set should benefit players. However while Φ monotonicity requires that *an enlargement of the feasible set* weakly increases the final allocation of any player, δ monotonicity only requires that for some step function δ associated with Φ *an enlargement of* *the feasible set* weakly increases the allocation of any player after the first interim agreement. More precisely there exists a δ associated with Φ such that if $S \subseteq T$, then the payoff $\delta_i(S, \mathbf{d})$ assigned to player i after the first interim agreement when the bargaining set is S is weakly less than the payoff $\delta_i(T, \mathbf{d})$ assigned to player i after the first interim agreement when the bargaining set is T. Using this axiom, we now state the main result of this section:

Proposition 3.2.2 ((λ, p)-Raiffa solutions - monotonicity)**.**
If Φ satisfies (IIR), (SI), (PO), (δ-MON) then Φ is a (λ, p)-weighted Raiffa solution.

The result shows that weighted Raiffa solutions are the only bargaining solutions that satisfy a monotonicity condition on a path of interim agreements. First note that the weighted Raiffa solutions satisfy the axioms: clearly (IIR), (SI) and (PO) are satisfied. To see (δ-MON) is also satisfied consider the following interim solution:

$$\delta_i(S, \mathbf{d}) = p\lambda_i m_i(\mathbf{d}|S) + (1 - p\lambda_i)\mathbf{s}_i \quad \text{for all} \quad i$$

Showing that no other solutions satisfy these axioms is non-trivial. The proof first considers a weaker axiom, which says that if the maximum gain available to player i - namely $m_i(\mathbf{d}|S)$ - increases as the bargaining set changes from S to T, then the allocation player i will receive after the first interim agreement will also increase:

δ-initial gain (δ-INITIAL) For some interim solution δ associated with Φ:

$$\text{If} \;\; m_i(\mathbf{d}|S) \leq m_i(\mathbf{d}|T) \;\; \text{then} \;\; \delta_i(S, \mathbf{d}) \leq \delta_i(T, \mathbf{d}) \text{ for all } \mathbf{d} \in S$$

Since $S \subset T$ implies $m_i(\mathbf{d}|S) \leq m_i(\mathbf{d}|T)$ it follows that δ-initial gain is implied by δ-monotonicity. Hence to prove the previous proposition, it is sufficient to prove the following lemma:

Lemma 3.2.3 ((λ, p)-Raiffa solutions - initial gain)**.**
If Φ satisfies (SI), (PO), (δ-INITIAL) then Φ is a (λ, p)-weighted Raiffa solution.

Since Raiffa solutions - unlike other bargaining solutions - determine interim agreements solely using the current interim agreement which can be interpreted as the current default allocation $\mathbf{d}$ and the current ideal point $m(\mathbf{d}|S)$ this result seems intuitively plausible. The proof builds on techniques from Diskin et al. (2011), who axiomatize the symmetric version. However it is by no means a simple extension, and additional complications arise here for two reasons: first dropping the symmetry axiom allows a wider class of weighted bargaining solutions to be axiomatized; secondly the Φ-scale invariance axiom here is weaker than the δ-scale invariance axiom used by Diskin et al. (2011).[3] This complicates the proof, but makes comparisons with other axiomatizations easier and clarifies the exact role of the interim agreement function. The next section explores such a comparison between this family of Raiffa solutions and the Kalai-Smorodinsky solution.

3.2.1 Weighted Kalai-Smorodinsky solutions

In this section we provide a new axiomatization for the family of weighted Kalai-Smorodinsky (KS) solutions. Unlike existing results - such as that found in Dubra (2001) - the axiomatization below appeals only to the concept of monotonicity. These solutions are defined as follows:

Definition 22. *Given a bargaining problem $(S, \boldsymbol{d})$ and bargaining weights λ the weighted KS solution is uniquely defined as follows:*

$$KS_\lambda(S,\boldsymbol{d}) = \left\{ \boldsymbol{s} | s_i = d_i + a\lambda_i \Big(m_i(S,\boldsymbol{d}) - d_i \Big) \ \ \textit{for all} \ \ i \cap WP(S) \ \ \textit{where} \ \ a \in \mathrm{Re}_+ \right\}$$

In order to motivate the following two axioms, we introduce the concept of *relative bargaining strength.* Given a bargaining problem $(S, \mathbf{d})$ we make the following definition:

$$R_{i,j}(S,\mathbf{d}) = \frac{m_i(S,\mathbf{d}) - d_i}{m_j(S,\mathbf{d}) - d_j}$$

For any bargaining problem $(S, \mathbf{d})$ the value $R_{i,j}(S, \mathbf{d})$ is referred to as the *relative bargaining strength* of player i to player j. This value is given by the ratio between the maximum gains player i can hope to achieve to the maximum gains player j can hope to achieve. We say that a change in the bargaining set from S to T

[3]In particular the δ-invariance used by Diskin et al. (2011) immediately implies Φ-invariance, while Φ-invariance does not imply δ-invariance.

places player i in a relatively stronger bargaining position compared with another player j if $R_{i,j}(S,\mathbf{d}) \leq R_{i,j}(T,\mathbf{d})$. An *enlargement of the bargaining set* also puts player i in a *relatively stronger bargaining position compared to another player j*, if the maximum gains player i can hope to achieve has increased proportionally more compared to the maximum gains player j can hope to achieve.

This intuition motivates another way of restricting that the Φ-monotonicity axiom. The alternative restriction says that if the *initial bargaining strength* of player i weakly increases then either a player's final allocation weakly increases or the relative bargaining strength of player i compared to some other player j strictly decreases. This leads to the following axiom:

Φ- initial monotonicity (Φ-R-INITIAL)
Suppose $m_i(\mathbf{d}|S) \leq m_i(\mathbf{d}|T)$. Then:

$$\text{Either} \quad \Phi_i(S,\mathbf{d}) \leq \Phi_i(T,\mathbf{d}) \quad \text{or} \quad R_{i,j}(S,\mathbf{d}) > R_{i,j}(T,\mathbf{d}) \quad \text{for some} \quad j$$

Combined with Pareto optimality, scale invariance and symmetry this axiom uniquely characterises the KS-solution. Moreover it is strictly weaker than the monotonicity axiom introduced by Kalai (1977).[4] However dropping the symmetry axiom does not lead to a unique family of weighted bargaining solutions: indeed the large class of monotonic solutions that satisfy the remaining three axioms has been fully characterised by Peters and Damme (1991). Therefore a characterisation of the family of KS weighted solutions requires an additional axiom.

The new axiom we propose uses a similar restriction to the one imposed on the Φ-monotonicity axiom above, but the restriction is placed on the weaker δ-monotonicity axiom. Although the δ-monotonicity axiom is attractive - the same interim agreement $\delta(S,\mathbf{d})$ may not always be appropriate for two bargaining problems $(S,\mathbf{d})$ and $(T,\mathbf{d})$ where players have the same *initial bargaining strength*. In particular the same interim agreement may not be appropriate in those cases where

[4]The monotonicity axiom introduced by Kalai (1977) is slightly stronger. Using the notation above, the first clause becomes $S \subset T$ and the second clause becomes: Either $\Phi_i(S,\mathbf{d}) \leq \Phi_j(S,\mathbf{d})$ or $m_j(S,\mathbf{d}) < m_j(T,\mathbf{d})$. We use the slightly weaker axiom above in order to show a connection with the new axiom introduced below

players initially have the same *initial bargaining strength* but when the agreement $\delta(S, \mathbf{d})$ leads to one player having much greater *relative bargaining strength* under the new problem $(S, \delta(S, \mathbf{d}))$ compared to the new problem $(T, \delta(S, \mathbf{d}))$. In such cases the player who is moving to a position with less *relative bargaining strength* may demand a higher allocation, to compensate for his loss in *relative bargaining power*. This motivates the following axiom:

δ-restricted monotonicity (δ-R-MON)
Suppose $m_i(S, \mathbf{d}) \leq m_i(T, \mathbf{d})$. Then there exists a non-trivial δ associated with Φ such that:

$$\text{Either} \quad \delta_i(S, \mathbf{d}) \leq \delta_i(T, \mathbf{d}) \quad \text{or} \quad R_{i,j}(S, \delta(S, \mathbf{d})) < R_{i,j}(T, \delta(S, \mathbf{d})) \quad \text{for some} \quad j \neq i$$

This axiom says that if the *initial bargaining strength* of player i (weakly) improves after a change from S to T then either this player will have a weakly higher allocation $\delta_i(T, \mathbf{d})$ after an interim agreement or the relative bargaining strength of player i compared to player j improves when the default is $\delta(S, \mathbf{d})$ and the bargaining set changes from S to T. This additional requirement allows a player to receive less after the first interim agreement in those cases when his *initial bargaining strength* has improved but where the same interim agreement $\delta(S, \mathbf{d})$ would result in a significantly greater *relative bargaining strength.* This extra restriction models the fact that players may take into account their future relative bargaining strength when deciding upon interim agreements: a player may be more reluctant to make an agreement if it leads to a situation where he is in a relatively weak bargaining position.

Since δ-restricted monotonicity is weaker than δ-monotonicity, it follows from above that weighted Raiffa solutions satisfy this axiom. It is shown in the appendix that weighted Kalai-Smorodinsky solutions also satisfy this axiom. Using the Φ-restricted monotonicity axiom above leads to the following characterisation:

Proposition 3.2.4 (Weighted Kalai-Smorodinsky solution)**.** *If Φ satisfies (SI), (PO), (Φ-RMON) and (δ-RMON), then Φ is a λ-weighted Kalai-Smorodinsky solution.*

Table 3.1. Axiom summary

	Generalized Raiffa	Kalai-Smorodinsky	Egalitarian	Nash
Pareto Optimality	✔	✔	✔	✔
Scale Invariance	✔	✔		✔
δ-monotonicity	✔		(✔)	
Restricted δ-monotonicity	(✔)	4*	(✔)	
Φ-monotonicity			✔	
Restricted Φ-monotonicity		✔	(✔)	
IIA			(✔)	✔
Restricted IIA		4**	(✔)	(✔)

Another axiomatization of the weighted family of KS solutions is provided by Dubra (2001). This axiomatization is based on Φ-monotonicity and a weakened version of the irrelevant alternatives axiom (IIA) used by Nash. In contrast here we show that weighted KS solutions can also be characterised by using a weakened version of δ-monotonicity rather than a weakened version of (IIA). A full summary of these results is given in Table 1. The unbracketed entries are sufficient to characterise the relevant family of weighted solutions. Meanwhile the bracketed entries are not needed for the characterisation, but are further properties satisfied by the solution in question. The starred entries capture the two alternative ways to axiomatize the weighted Kalai-Somorodinsky solution:

This shows the strong connection between the cooperative foundations of weighted Raiffa solutions and weighted KS solutions. Both solution families can be characterised by appealing to the concepts of monotonicity, scale invariance and Pareto optimality. On the one hand, weighted KS solutions are more forward-looking and focus primarily on the eventual outcome. This family can can be characterised using restricted monotonicity axioms on both the bargaining solution as well as the path of interim solutions. In contrast the weighted Raiffa solutions put more focus on the current default point, and can be characterised by a stronger monotonicity axiom on the path of interim solutions.

3.3 Non-cooperative foundation

We now consider non-cooperative foundations for (λ, p)-Raiffa solutions, by showing how these solutions can arise from simple non-cooperative games. More precisely we suggest bargaining procedures that could be used by a planner to implement a certain (λ, p)-Raiffa solution, in situations where the set of utility functions $(u_i)_{i \in N}$ is common knowledge among players but is not known by the planner. The procedures we consider implement bargaining solutions to any arbitrary degree of accuracy. Exact implementation can be achieved by modifying the procedure considered by Trockel (2011).

First we outline the bargaining procedure. Without loss of generality, consider bargaining sets S where the default is normalized to $\mathbf{d} = \mathbf{0}$. Players have T rounds to reach an agreement. In each round the bargaining procedure may be in one of $(n+1)$ states. When the procedure is in state i, player i makes an offer, while if the process is in state $(n+1)$ no offer is made. The player selected to make an offer proposes a feasible allocation. If all other players accept the game ends and the allocation proposed is implemented. Otherwise the game continues to the next round. If after T rounds no agreement is reached, then the default allocation $\mathbf{0}$ is implemented.

In the first round the bargaining procedure starts in state 1. In every subsequent round the state evolves according to a Markov process, with transition matrix Q. Let q_{jk} be the probability that given that negotiations are in state j in round t, the state in round $t+1$ is k.

We first introduce some additional notation. When τ rounds remain and negotiations are in state j we define $r_i^{\tau,j}$ for each player i as follows:

- When no rounds remain: $r_i^{0,j} = \mathbf{0}$ for all players i and all states j
- When τ rounds remain: $r_i^{\tau,j} = q_{ik} m_i(r^{\tau-1,i}, S) + \sum_{j \neq i} q_{jk} r_j^{\tau-1,k}$ for all players i and all states j

When it is clear from the context which bargaining problem (S, d) is being referred to, we abuse notation by writing $\hat{m}^{\tau,i} = m_i(r^{\tau,i}, S)$. Moreover we write (a_i, b_{-i}) to refer to a vector with the i'th element equal to a_i and the j'th element equal to b_j whenever $i \neq j$. We first prove a preliminary lemma:

Lemma 3.3.1.
The utility allocation $r^{\tau,j}$ is in the feasible set: $r^{\tau,j} \in S$ for all j and for all τ

Proof. The proof follows by induction. The base case is trivial, since $r^{0,j} = \mathbf{0} \in S$. For the inductive step assume $r^{\tau,j} \in S$. Define $s^{\tau,i} = (\hat{m}^{\tau,i}, r^{\tau,i}_{-i})$ and note that $s^{\tau,i} \in S$. Using the inductive assumption we get:

$$\begin{aligned} r_i^{\tau+1,j} &= \sum_{k \neq i} q_{jk} r_i^{\tau,k} + q_{ji} \hat{m}^{\tau,i} \\ &= \sum_{k \neq i} q_{jk} s^{\tau,k} + q_{ji} s^{\tau,i} \end{aligned}$$

Hence $r^{\tau+1,j} = \sum_{k \in N} q_{j,k} s^{\tau,k}$. Since all vectors $s^{\tau,k} \in S$ and S is a convex set, it follows that $r^{\tau+1,j} \in S$ □

Using this lemma, we now show that the following is a subgame perfect equilibrium (SPE):

Proposition 3.3.2 (SPE with immediate acceptance)**.**
The bargaining model with transition matrix Q has the following SPE. If τ additional rounds remain before the default is implemented and negotiations are in state $i \leq n$, then player i proposes $(\hat{m}^{\tau,i}, r^{\tau,i}_{-i})$. This offer is accepted by all other players.

The proof follows by induction and can be found in the appendix, where we show that players have no profitable deviations in any subgame.

Here $r_i^{\tau,j}$ is player i's expected utility of continuing negotiations, and the minimum utility level he is prepared to accept. The proposer j offers each player i this reservation utility $r_i^{\tau,j}$, while assigning himself the remainder $m_j(r^{\tau,j}, S)$. Moreover the proposer at least weakly prefers this allocation to continuing to the next round, while the other players are indifferent.

If player 1 is strictly risk averse and receives a strictly positive utility allocation $\hat{m}_1^{\tau,1} > 0$, then this is the unique SPE. This is because risk averse proposers strictly prefer to make an acceptable offer rather than continuing to the next round.[5]. Intuitively this result follows from the fact that delaying agreement until close to the deadline increases the risk a player faces and hence a risk averse proposer will strictly prefer to make an immediate agreement. If player 1 is risk neutral or $\hat{m}_1^{\tau,1} = 0$, then there may be other SPEs where player 1 makes an unacceptable initial offer. However these (unusual) SPEs lead to the same expected utility as the one defined above. Therefore we restrict attention to the equilibrium characterised above. We now consider a specific family of transition matrices $Q_0(\lambda, p)$, where $q_{i,j} = p\lambda_j$ whenever $1 \leq j \leq n$ and $q_{i,n+1} = 1 - p$. The two player case is given by:

$$Q_0(\lambda, p) = \begin{pmatrix} p\lambda_1 & p\lambda_2 & 1-p \\ p\lambda_1 & p\lambda_2 & 1-p \\ p\lambda_1 & p\lambda_2 & 1-p \end{pmatrix} \tag{3.1}$$

This transition matrix models a situation where offers are made intermittently. In every round each player has a fixed chance of being selected to be the proposer. Using this simple transition matrix where all the rows are the same, leads to the following result. All remaining proofs an be found in the appendix.

Proposition 3.3.3 (Intermittent Offers).
Consider the bargaining model with transition matrix $Q_0(\lambda, p)$, where proposals are made intermittently. Assume players follow the SPE strategies outlined above. As the number of rounds $T \to \infty$, the utility players obtain converge to their utility level under the (λ, p)-weighted Raiffa solution.

We now consider bargaining models where proposals are made regularly in every round. However instead of each player having a fixed chance of being the proposer in each round, we assume that the player who proposed in round t is more likely to propose again in round $t + 1$. This captures situations where one party is able to revise their offer an uncertain number of times before another party can make a proposal. To model these situations with persistent offers, we define the transition

[5] Meanwhile no SPE exists where player j rejects a proposal when indifferent. This is because in this case the proposer would maximize his utility by proposing the lowest $s_j > r_j^{\tau,1}$

matrix
$Q_1(\lambda, p) := pQ_0(\lambda, 1) + (1-p)I$. In the two player case:

$$Q_1(\lambda, p) = \begin{pmatrix} p\lambda_1 + 1 - p & p\lambda_2 & 0 \\ p\lambda_1 & p\lambda_2 + 1 - p & 0 \\ p\lambda_1 & p\lambda_2 & 1-p \end{pmatrix} \tag{3.2}$$

This model with persistent offers leads to the following result:

Proposition 3.3.4 (Persistent offers)**.**
Consider the bargaining model with transition matrix $Q_1(\lambda, p)$, where the identity of the proposer is persistent. Assume players follow the SPE strategies outlined above. As the number of rounds $T \to \infty$, the expected utility players obtain converge to their utility level under the (λ, p)-weighted Raiffa solution.

Hence the family of weighted Raiffa solutions can be implemented either when offers are made intermittently, or when the identity of the proposer is persistent. The theorem below shows that these solutions can also be implemented by models combining both these features:

Theorem 3.3.5.
Consider any model with transition matrix $Q_\mu(\lambda, p) = \mu Q_0(\lambda, p) + (1-\mu)Q_1(\lambda, p)$ where $\mu \in [0, 1]$. Assume players follow the SPE strategies outlined above. As the number of rounds $T \to \infty$, the utility players obtain converge to their utility level under the (λ, p)-weighted Raiffa solution.

Proof. Define $d(0) = 0$ and $d_i(\tau) := (1 - \lambda_i p) + \lambda_i p m_i(S, d(\tau - 1))$. Note that $d(\tau)$ converges to the (λ, p)-Raiffa solution, as $\tau \to \infty$. After appealing to the immediate acceptance lemma, it remains to be shown that for all τ:

$$d_i(\tau) = r_i^{\tau,j} \;\; \forall j \neq i$$

We prove this by induction. The base case is trivial. From their respective definitions, $r^{0,j} = d(0) = 0$ for all j.
Now consider the inductive step. Suppose $d_i(\tau) = r_i^{\tau,j} \; \forall \; j \neq i$.

$$
\begin{aligned}
r_i^{\tau+1,j} &= \mu\Big(\sum_{k=1,k\neq i}^{n} \lambda_k p r_i^{\tau,k} + \lambda_i p \hat{m}_i^{\tau,i} + (1-p) r_i^{\tau,n+1}\Big) \\
&\quad +(1-\mu)\Big(\sum_{k=1,k\neq i}^{n} \lambda_k p r_i^{\tau,k} + \lambda_i p \hat{m}_i^{\tau,i} + (1-p) r_i^{\tau,j}\Big) \\
&= \sum_{k=1,k\neq i}^{n} \lambda_k p r_i^{\tau,k} + (1-p) r_i^{\tau,n+1} + (1-\mu)(1-p)) r_i^{\tau,j} + \lambda_i p m_i(S, r^{\tau,i}) \\
&= (1-\lambda_i p) d_i(\tau) + \lambda_i p m_i(S, d(\tau)) \\
&= d_i(\tau+1)
\end{aligned}
$$

The first line states the definition of $r_i^{\tau+1,j}$. Rearranging and collecting terms leads to the second line. The third line uses the induction hypothesis, using the fact that reservation utilities are the same in every state. In the final step, the definition of d is applied. Since $m_i(S, d(\tau) \geq d_i(\tau)$ it follows that when τ rounds remain, all players receive at least $d_i(\tau)$. Hence as the number of rounds remaining $\tau \to \infty$, the utility players obtain converge to $d(\tau)$. Therefore if T is large, then the (λ, p)-Raiffa solution is approximately implemented. □

These implementations show that the (λ, p)-Raiffa solution arises from a number of simple bargaining models, where offers are made intermittently or the identity of the proposer is persistent. In particular any (λ, p)-Raiffa solution can be implemented by a simple bargaining model without discounting, where in every round players have a fixed chance of being the proposer regardless of past history. This provides a strong non-cooperative foundation for this family of weighted bargaining solutions.

3.4 Discussion

When players are risk neutral, the bargaining weights correspond to the proportion of cake each player is allocated in equilibrium as is the case under the Nash and the KS solution. Hence the solution concepts coincide when players are risk neutral.

We have shown that the (λ, p)-Raiffa solution can be implemented in simple games using discrete time. When players have equal bargaining weights and $p \to 0$ which means that offers are made rarely, the model presented can approximately implement the continuous Raiffa solution. In the two player case, this is done using the following Q-matrix, where $\epsilon \to 0$.

$$Q^{\epsilon}(\lambda, p) = \begin{pmatrix} \epsilon & \epsilon & 1-2\epsilon \\ \epsilon & \epsilon & 1-2\epsilon \\ \epsilon & \epsilon & 1-2\epsilon \end{pmatrix}$$

This discrete time model is related to the continuous time model studied by Ambrus and Lu (2015a), who consider offers arriving according to a Poisson process. This suggests that the λ-continuous Raiffa solution and related weighted solutions may also arise in continuous time settings.

Finally the implementations introduced have similarities with implementations of the weighted Nash solution. In particular Britz et al. (2010) show that a weighted Nash solution arises in settings similar to those considered above, where there are infinitely many rounds and players discount at a rate tending to one. We conjecture that if a discount factor is added to the finite horizon game described above, the weighted Nash solution will be implemented whenever $\beta \to 1$, $\beta^T \to 0$ and Q is irreducible. This result would bridge the infinite horizon result proved in Britz et al. (2010) and the finite horizon result proved here.

Currently the weighted Nash solution is the standard solution used in applications by economists. However we argue here that the weighted (λ, p)-Raiffa solutions also have strong foundations and may in some situations be preferred, particularly in settings where discounting is unimportant.

3.5 Appendix

3.5.1 Literature Summary:

Table 3.2. Literature summary

		Nash		Kalai-Smorodinsky		Raiffa $p = \{0, 1\}$		Raiffa $p = (0, 1)$	
SYMMETRIC	Axiomatic	Nash	(1953)	Kalai-Smorodinsky	(1970)	Raiffa	(1953)	Diskin et al	(2011)
VERSION	Simultaneous	Nash	(1953)	Moulin	(1984)	-	-	-	-
	Sequential	Binmore et al	(1970)	-	-	Myerson Sjostrom $p = 1$ Gomes	(1991) (1991) (1998)	Diskin et al	(2011)
ASYMMETRIC	Axiomatic	Chung & Ely	(1975)	Dubra This paper	(2001)	This paper		This paper	
VERSION	Simultaneous	Carlsson	(1991)	-	-	-	-	-	-
	Sequential	Britz et al	(2011)	-	-	This paper		This paper	

3.5.2 Additional notation

We first introduce some additional notation. Throughout the appendix we use bold numbers to refer to a vector with n elements all of which are equal to the bold number. Hence $\mathbf{1}$ represents the unit vector. We use $\mathbf{k} \in \mathrm{Re}^n$ to refer to a generic vector such that $\mathbf{k}_i = k$ for all i. Moreover a bold letter $\mathbf{v}$ is used to refer to any generic vector $\mathbf{v} = (v_1, \ldots, v_n) \in \mathrm{Re}^n_{++}$.

3.5.3 Consequences of scale invariance

Define a such that $a\mathbf{1} = \mathbf{a} \in PF(S)$ and define $\hat{\Delta}(\lambda) = CH\{\mathbf{0}, (\frac{1}{\lambda_i}, \mathbf{0}_{-i})\}$. Using these definitions let $T_{a,\lambda} = CH(\hat{\Delta}(\lambda), \mathbf{a})$. We first prove a lemma related to these convex sets. It says that if the process starts on a path towards $\mathbf{a}$ then it does not change direction.

Lemma 3.5.1. *Suppose Φ is associated with a partial solution δ which satisfies δ-monotonicity. Moreover suppose $\delta(T_{a,\lambda}, \mathbf{0}) = p\mathbf{a}$. Then $\Phi(T_{a,\lambda}, \mathbf{0}) = \boldsymbol{a}$.*

Proof. Consider the following linear mapping M: $\lambda \mapsto \lambda$ and $\delta(S, \mathbf{0}) \mapsto \mathbf{0}$. Now note that $M(S_{\delta(S,\mathbf{0})}) = S$. Hence by scale invariance:

$$\begin{aligned} M\Big(\Phi(S_{\delta S,d}, \delta(S, \mathbf{0}))\Big) &= \Phi(S, \mathbf{0}) \\ M\Big(\Phi(S, \mathbf{0})\Big) &= \Phi(S, \mathbf{0}) \end{aligned}$$

But since the linear mapping M has a unique fixed point, given by λ, it follows that: $\Phi(S, \mathbf{0}) = \lambda$.

□

Lemma 3.5.2. *If $\Phi(\Delta(\lambda), \mathbf{0}) = \mathbf{1}$, then $\delta(\Delta(\lambda), \mathbf{0}) = p\mathbf{1}$*

Proof. Let $\delta(\Delta(\lambda), \mathbf{0}) = p\mathbf{a}$ where $\mathbf{a} \in PF(\Delta(\lambda))$. By the lemma above $\Phi(\Delta(\lambda), \mathbf{0}) = \mathbf{a}$. Hence $\mathbf{a} = 1$ and $\delta(\Delta(\lambda), \mathbf{0}) = p\mathbf{1}$. □

3.5.4 Proof: Weighted Raiffa solution

We split the proof of proposition 3.2.2 into two parts for clarity. The second part is similar to the proof of the symmetric case which can be found in Diskin et al. (2011). However since the axioms considered here are weaker - in particular the Φ-invariance axiom is weaker than the δ-invariance axiom used by Diskin et al. (2011), a longer proof is required.

First the simplex is defined as follows:

$$\Delta(\mathbf{1}) = \{\mathbf{s} \in \mathrm{Re}^n \,|\, \sum_{i=1}^{n} s_i \leq 1 \text{ and } \mathbf{s} \geq 0\}$$

For any vector $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathrm{Re}^n_{++}$ the invertible linear transformation $f^{\mathbf{v}} : \mathrm{Re}^n \mapsto \mathrm{Re}^n$ is defined as follows: for any $\mathbf{s} \in \mathrm{Re}^n$, $f_i^{\mathbf{v}}(\mathbf{s}) = v_i s_i$. Using the linear transformation $f^{\mathbf{v}}$ we can now define the stretched simplex $\Delta(\mathbf{v})$ as follows:

$$\Delta(\mathbf{v}) = f^{\mathbf{v}}(\Delta(\mathbf{1}))$$

In the first section of the proof, we show that for some $p \in [0,1]$ and some $\lambda \in \mathrm{Re}^n_{+}$, $\frac{1}{k}\delta(\Delta(\mathbf{k}), 0) \to p\lambda$ as $k \to \infty$. We can then use this fact in the second part of the proof and avoid using the stronger axiom of δ-invariance.

Lemma 3.5.3. *$\frac{1}{k}\delta(\Delta(\boldsymbol{k}), \boldsymbol{0}) \to p\lambda$ as $k \to \infty$ for some $\lambda \in [0,1]^n$ and $p \in (0,1]$.*

Proof. Note that $\delta(\Delta(\mathbf{v}), \mathbf{0}) = g(\mathbf{v}) f^{\mathbf{v}}(\lambda)$ where $g(\mathbf{v}) \in \mathrm{Re}$ is a constant multiplying each element in $f^{\mathbf{v}}(\lambda)$. If not it follows from Φ-invariance that $\Phi(\Delta(\mathbf{v}), \mathbf{0}) \neq \Phi(\Delta(\mathbf{v}), \delta(\Delta(\mathbf{v}), \mathbf{0}))$ and this violates the fact that δ is associated with Φ.

Note that $f^{\mathbf{k}}(\lambda) = k\lambda$ and hence $\frac{1}{k}\delta(\Delta(\mathbf{k}), \mathbf{0}) = g(\mathbf{k})\lambda$.

Take $k' > k$ and define $\mathbf{v} = (k'_1, k_{-1})$. Now $\Delta(\mathbf{k}) \subset \Delta(\mathbf{v}) \subset \Delta(\mathbf{k'})$, and so by δ-monotonicity $\delta(\Delta(\mathbf{k}), \mathbf{0}) \leq \delta(\mathbf{v}, 0) \leq \delta(\Delta(\mathbf{k'}), \mathbf{0})$.

Using the definition above, it follows that $g(\mathbf{k})f^{\mathbf{k}}(\lambda) \leq g(\mathbf{v})f^{\mathbf{v}}(\lambda) \leq g(\mathbf{k}')f^{\mathbf{k}'}(\lambda)$. Now the first inequality implies that $g(\mathbf{k}) \leq g(\mathbf{v})$. Suppose this was not true. Then, since $f_n^{\mathbf{v}} = f_n^{\mathbf{k}}$ the last component of $\delta(\Delta(\mathbf{k}), \mathbf{0})$ is higher than the last component of $\delta(\Delta(\mathbf{v}), \mathbf{0})$. This violates δ-monotonicity. Meanwhile the second inequality implies that $g(\mathbf{v}) \leq g(\mathbf{k}')$. Suppose this was not true. Then since $f_1^{\mathbf{v}} = f_1^{\mathbf{k}'}$, the first component of $\delta(\Delta(\mathbf{v}), \mathbf{0})$ is higher than the first component of $\delta(\Delta(\mathbf{k'}), \mathbf{0})$. This violates δ-monotonicity.

Putting these inequalities together implies $g(\mathbf{k}) \leq g(\mathbf{v}) \leq g(\mathbf{k}')$. Note that $g(\mathbf{k})$ is increasing in each of its arguments. Moreover $g(\mathbf{k})$ is bounded above by 1, since otherwise $\delta(\Delta(\mathbf{k}), 0) \notin \Delta(\mathbf{k})$. Furthermore $g(\mathbf{k})$ must be positive, or $\delta(\Delta(\mathbf{k}), \mathbf{0}) \notin \Delta(\mathbf{k})$, which violates the requirement that δ is feasible. Finally $g(\mathbf{k}) \neq 0$, because otherwise repeated applications of the interim agreement function would not change the interim agreement point and hence the pareto optimal solution would never be reached. Hence $g(\mathbf{k}) \in (0, 1]$ and is increasing in each of its elements k_i for ß $\in \{1, \ldots, n\}$.

Therefore as $k \to \infty$ and considering the particular sequence of $\mathbf{k}$, it must be the case that for some $p \in (0, 1]$, $g(\mathbf{k}) \to p$. Since $\delta(\Delta(\mathbf{k}), \mathbf{0}) = g(\mathbf{k})f^{\mathbf{k}}(\lambda)$ and $f^{\mathbf{k}}(\lambda) = k\lambda$, it follows that $\frac{1}{k}\delta(\Delta(\mathbf{k}), \mathbf{0}) \to p\lambda$.

□

Lemma 3.5.4. *If $\mathbf{0} \in S$ and $m(S, \mathbf{0}) = \mathbf{1}$, then $\Phi(S, \mathbf{0}) = \Phi(S, p\lambda)$*

Proof. Take any normalised bargaining problem $(S, \mathbf{0})$ such that the ideal point $m(\mathbf{0}) = \mathbf{1}$. In order to show the result, it is sufficient to prove that for any degree of accuracy $\bar{\epsilon} > 0$ there exists an interim agreement d^1 such that $||d^1 - p\lambda|| < \bar{\epsilon}$ and $\Phi(S, \mathbf{0}) = \Phi(S, d^1)$.

First - using the earlier result - pick K such that for all $k > K$, $(p - \epsilon)\lambda \leq \frac{1}{k}\delta(\Delta(\mathbf{k}), \mathbf{0}) \leq p\lambda$. Define $S(k) = f^{\mathbf{k}}(S)$ and consider the stretched simplex $\Delta(\mathbf{k}) \subseteq S(k)$. By δ-monotonicity, $\delta(\Delta(\mathbf{k}), \mathbf{0}) \leq \delta(S(k), \mathbf{0})$. Since $\frac{1}{k}\delta(\Delta(\mathbf{k}), \mathbf{0}) \geq (p - \epsilon)\lambda$, it follows that $\delta(S(k), \mathbf{0}) \geq k(p - \epsilon)\lambda$.

Secondly for $a \in \text{Re}$ where $a > 2$ define $\mathbf{v}(a,i)$ as follows. $\mathbf{v}_i(a,i) = \frac{ak}{a-1}$ and whenever $j \neq i$ $\mathbf{v}_j(k,i) = ak$. Note that for some scalar $b \in \text{Re}_+$, $\delta(\Delta(\mathbf{v}(a,i)), \mathbf{0}) = bf^{\mathbf{v}(a,i)}(\lambda)$. If not the bargaining solution would violate Φ-invariance, or the interim agreement function δ would not be associated with Φ.

Now note that $\Delta(\mathbf{v}(a,i)) \subseteq \Delta(\mathbf{ak})$. Using δ-monotonicity for all $j \neq i$:

$$\begin{aligned}
\delta_j(\Delta(\mathbf{v}(a,i)), \mathbf{0}) &= bf_j^{\mathbf{v}(a,i)}(\lambda) \\
&= bak\lambda_j \\
&\leq \delta_j(\Delta(\mathbf{ak}), \mathbf{0}) \\
&\leq pak\lambda_j
\end{aligned}$$

It follows that $b \leq p$. But also note that $S(k) \subseteq \Delta(\mathbf{v}(a,i))$, and so by δ-monotonicity:

$$\begin{aligned}
\delta_i(\Delta(\mathbf{v}(a,i)), \mathbf{0}) &= bf_i^{\mathbf{v}(a,i)}(\lambda) \\
&= \frac{bak}{a-1}\lambda_i \\
&\geq \delta_i(S(k), \mathbf{0})
\end{aligned}$$

Since a can be arbitrarily high, it follows that $\delta(S(k), \mathbf{0}) \leq bk\lambda \leq pk\lambda$.

Finally define $d^1 = \frac{1}{k}\delta(S(k), \mathbf{0})$. Using Φ-invariance twice note that:

$$\begin{aligned}
f^{\mathbf{k}}\Big(\Phi(S,\mathbf{0})\Big) &= \Phi(f^{\mathbf{k}}(S), f^{\mathbf{k}}(\mathbf{0})) \\
&= \Phi(S(k), \mathbf{0}) \\
&= \Phi(S(k), \delta(S(k), \mathbf{0})) \\
&= \Phi(S(k), kd^1) \\
&= \Phi(f^{\mathbf{k}}(S), f^{\mathbf{k}}(d^1)) \\
&= f^{\mathbf{k}}\Big(\Phi(S, d^1)\Big)
\end{aligned}$$

Since $f^{\mathbf{k}}$ is invertible, this implies that $\Phi(S,\mathbf{0}) = \Phi(S, d^1)$. Also note from the two results above that $k(p-\epsilon)\lambda \leq \delta(S(k),\mathbf{0}) \leq kp\lambda$, and hence $(p-\epsilon)\lambda \leq d^1 \leq p\lambda$. Since ϵ is arbitrarily small, this proves the result. □

3.5.5 Proof: Weighted Kalai-Smorodinsky solution

Proof: 3.2.4 Weighted Kalai-Smorodinsky solution

Proof. Take a normalised bargaining problem $(S, \mathbf{d})$ such that:

1. $\mathbf{d} = \mathbf{0}$
2. $m_i(S, \mathbf{0}) = \frac{1}{\lambda_i}$

Define a such that $a\mathbf{1} = \mathbf{a} \in PF(S)$. Moreover define $\hat{\Delta}(\lambda) = CH\{\mathbf{0}, (\frac{1}{\lambda_i}, \mathbf{0}_{-i})\}$. Note that $\Phi(\Delta, \mathbf{0}) = \lambda$ and hence by - scale invariance - $\Phi(\hat{\Delta}(\lambda), \mathbf{0}) = \mathbf{1}$.

Finally define $T := CH\{S, \mathbf{a}\}$. Appealing to the two lemmas, for some scalars $p > 0$ and $p' > 0$:

$$\begin{array}{ccc}
\Phi(\hat{\Delta}(\lambda), \mathbf{0}) & = & \mathbf{1} & \text{(scale invariance argument above)} \\
\delta(\hat{\Delta}(\lambda), \mathbf{0}) & = & \text{p } \mathbf{1} & \text{(using lemma 3.5.2)} \\
\delta(T, \mathbf{0}) & = & \text{p' } \mathbf{1} & (\delta\text{-monotonicity lemma}) \\
\Phi(T, \mathbf{0}) & = & \mathbf{a} & \text{(using lemma 3.5.1)} \\
\Phi(S, \mathbf{0}) & = & \mathbf{a} & (\Phi\text{-monotonicity}) \\
\Phi(S, \mathbf{0}) & = & \Phi_\lambda^{KS}(S, \mathbf{0}) & (\text{definition of } \Phi_\lambda^{KS})
\end{array}$$

This proves that $\Phi(S, \mathbf{0}) = \Phi_\lambda^{KS}(S, \mathbf{0})$, whenever S is suitably normalised. The full result follows by appealing to scale invariance.

□

3.5.6 Proofs: Non cooperative foundation

Proof: 3.3.2 SPE

Proof. The proof follows by induction. We show that players have no profitable deviations in any subgame. Consider the base case. WLOG suppose negotiations are in state i in the last round.

1. Any player $j \neq i$ is indifferent between accepting (receiving $s_j = 0$) and rejecting (receiving $s_j = 0$). Hence rejecting is not a profitable deviation.

2. Suppose player i makes a lower offer, where for one player $s_j < 0$. In this case player j rejects and player i receives $d_i = 0 \leq m_i(S, \mathbf{0})$. Hence making a lower offer is not a profitable deviation.

3. Suppose player i makes a higher offer where $s_j > 0$ and $s_{-j} \geq 0$. The offer is accepted, but since, $m_i()$ is strictly decreasing in the utilities of other players $m_i(S, (\mathbf{0}_{-j}, x_j)) < m_i(S, \mathbf{0})$. Hence making a higher offer is not a profitable deviation.

Hence if negotiations are in state $i \leq n$ in the last round, then player i proposes $(\hat{m}^{0,i}, r_{-i}^{0,i})$ and this offer is accepted

Now, consider the inductive step. Suppose that when τ rounds remain player i proposes $(\hat{m}^{\tau,i}, r_{-i}^{\tau,i})$.

1. If player $j \neq i$ rejects, the process will move to the next round and will enter state k with probability $q_{i,j}$. By the inductive step, in the next round the offer $(\hat{m}^{\tau,k}, r_{-k}^{\tau,k})$ will be accepted. Hence if player j rejects he expects to receive $\sum_{k \neq j} q_{ik} r_i^{k,\tau} + q_{ij} m_i(r^{\tau,j}) = r_i^{\tau,j}$, So player $j \neq i$ is indifferent between accepting and rejecting. Hence rejecting is not a profitable deviation.

2. If player i makes any player a lower offer, where $s_j < r_j^{\tau,i}$ player j rejects. In this case - by using the same argument as above, player i expects to receive $r^{\tau,i}$ in the next round. Since $r^{\tau,i} \in S$, it follows that $\hat{m}^{\tau,i} \geq r^{\tau,i}$. Hence making a lower offer is not a profitable deviation.

3. If player i makes a higher offer $\mathbf{s}$ where $s_j > r_j^{\tau,i}$ and $s_{-i} \geq r_{-i}^{\tau,i}$, then the offer is accepted. However since $m_i()$ is strictly decreasing in the utilities of other players $m_i(S, \mathbf{s}) < m_i(S, r^{\tau,i})$. It follows that $s_i < \hat{m}^{\tau,i}$, and so making a higher offer is not a profitable deviation.

Hence if negotiations are in state $i \leq n$ with τ rounds remaining, then player i proposes $(\hat{m}^{\tau,i}, r_{-i}^{\tau,i})$ and this offer is accepted. $\square$

Chapter 4

A Non-Cooperative Foundation for the Continuous Raiffa Solution

4.1 Introduction

Almost simultaneously to the seminal work of Nash (1950), Nash (1953), Raiffa (1953) formulated two closely related alternatives to the Nash bargaining solution. The first – the *discrete* or *sequential* Raiffa solution – is defined as the limit of a series of intermediate agreements that is constructed by an iterative random dictator procedure. The second solution he proposed – the *continuous* Raiffa solution – is in the same spirit, but assumes that the step size between intermediate agreements is infinitesimally small. As such, it is obtained as the endpoint of a continuous intermediate agreement *curve*. The advantage of this second approach is that the resulting solution remains well-defined if bargaining problems are allowed to be non-convex. Indeed, while it is standard to assume that the feasible sets in the bargaining problem are convex, there is a wide literature that recognizes that this requirement may in many instances be too strict.[1] As such, we do not require that bargaining problems have convex feasible sets.

A prolific strand of literature in economics, known as the *Nash program*, focuses on non-cooperative foundations for cooperative solution concepts. More specif-

[1] See among others Conley (1991), Conley and Wilkie (1994), Conley and Wilkie (1996), Zhou (1997), Xu (2006), Xu and Yoshihara (2013)

ically, this agenda aims at constructing reasonable non-cooperative games such that their (unique) equilibrium payoffs yield, or somehow approximate, the outcome of the considered cooperative solution concept. While several results provide non-cooperative support for the discrete Raiffa solution, the continuous version has received much less attention.[2] With this paper we aim to address this gap in the literature. In particular, we provide a direct support result for the continuous Raiffa solution.

The game constructed to this end is an n-player bargaining model in the tradition of Stahl (1972) and Rubinstein (1982). Players are assumed to make proposals that are instantaneously accepted or rejected by all opponents; in case of unanimous agreement, the proposal is implemented, otherwise it is rejected and the game continues. Apart from these common features, the game differs from the classic models in several important ways: bargaining occurs in continuous time and the game features a finite deadline that ends the negotiations with all players obtaining zero payoffs. Moreover, the timing of the proposals is stochastic in the sense that they are governed by independent player-specific Poisson processes. It turns out that this game has a unique subgame perfect equilibrium (SPE) in which players use Markovian strategies, and the first proposal made is accepted. The main result of this paper is that the payoffs players realize in this SPE converge to the continuous Raiffa solution as the horizon tends to infinity.

Ambrus and Lu (2015b) consider a *coalitional* version of the above-described game, in the spirit of Okada (1996), Okada (2011), Chatterjee and K. (1993) and Yan (2003). The key distinction between the two versions lies in their respective underlying cooperative problems: Ambrus and Lu (2015b) take this to be a convex TU game, whereas the present paper assumes it is a pure bargaining problem. As is well-known, both are special cases of the more general class of NTU games (see a.o. Hart (2004)). Furthermore, Ambrus and Lu (2015b)'s framework assumes that players are impatient in the sense that utilities are discounted over time. These distinctions are important. Where we obtain non-cooperative support for the continuous Raiffa solution, as described above, Ambrus and Lu (2015b) obtain a non-cooperative foundation for the core of the underlying TU game.

[2] A notable exception is Diskin et al. (2011), who provide support for solutions that approximate the continuous Raiffa solution. See the discussion of the related literature below.

Finally, it is demonstrated that the support result does not depend in full on the chosen proposer protocol. In particular, we adopt a variation of the well-known *rejector-proposes* protocol, studied by Selten (1981), Chatterjee and K. (1993) and Britz and Predtetchinski (2012) among others, and show that this does not affect the SPE, nor the associated payoffs, nor its limit as the horizon tends to infinity.

Related Literature The Nash program literature on the Raiffa solution has primarily focused on its discrete version. Myerson (1991)(pp. 393-394) describes a two-player, discrete- and finite-time, random-recognition bargaining game that can be regarded a discrete-time analogue of the game considered in this paper. The payoffs associated with the unique SPE of this game converge to the *discrete* Raiffa solution, as the number of bargaining rounds T diverges to infinity.

Sjostrom (1991) proposes a similar game with the assumption that payoffs are discounted with factor r. This game too has a finite deadline that ends negotiations, and actions take place at T equidistant time points within this fixed time interval. Sjostrom (1991) demonstrates that the unique SPE payoffs of this game converge to an outcome within a certain distance from the discrete Raiffa solution, as the partition of the bargaining interval $[0, T]$ becomes more and more refined.

Diskin et al. (2011) introduce a class of *generalized* Raiffa solutions for n players; each such solution corresponds to the limit point of a series of intermediate agreements, where the step size between agreements lies within the interval $(0, 1/n]$. They provide a non-cooperative foundation for their solution class that is again based on Myerson's game. Of course, this support result only holds for generalized solutions with strictly positive step size, and thus necessarily entails an approximation that is absent from the model in this paper. To our knowledge this paper provides a first *direct* support result for the continuous Raiffa solution.

Structure of the Paper The remainder of the paper is organized as follows. Section 2 introduces the bargaining problem and the continuous Raiffa solution. Section 3 describes and analyzes the non-cooperative bargaining game. In Section 4 it is demonstrated that the payoffs associated with the unique SPE of this game converge to the continuous Raiffa solution as the horizon tends to infinity. Section 5 considers an alternative proposer protocol, and Section 6 concludes.

4.2 Preliminaries

4.2.1 The Bargaining Problem

A *bargaining problem* is defined by a finite set of players $N := \{1, \ldots, n\}$ with $n \geq 2$, and a subset S of $\mathbb{R}^n$, that is closed and strictly comprehensive (i.e. $y \in S$ and $x \leq y$ implies $x \in S$; if $x \neq y$, then $z > x$ for some $z \in S$)[3], that contains an outcome $z > \mathbf{0} =: (0, \ldots, 0)$, and is such that $S \cap \mathbb{R}^n_+$ is bounded. It is further assumed that S satisfies the following condition:[4]

(A1): There exists a $K > 0$ such that for all $i, j \in N$, and for all $x, y \in \partial S \cap \mathbb{R}^n_+$ with $x_{-\{i,j\}} = y_{-\{i,j\}}$: $|x_i - y_i| \leq K|x_j - y_j|$.

Note that we do not insist on convexity of S.

Fixing the set of players N, a bargaining problem is henceforth denoted by its feasible set S. The class of all bargaining problems S is denoted $\mathcal{B}$. A *bargaining solution* is a map $\varphi : \mathcal{B} \to \mathbb{R}^n$ that assigns to each bargaining problem $S \in \mathcal{B}$ a unique outcome $\varphi(S) \in S$.

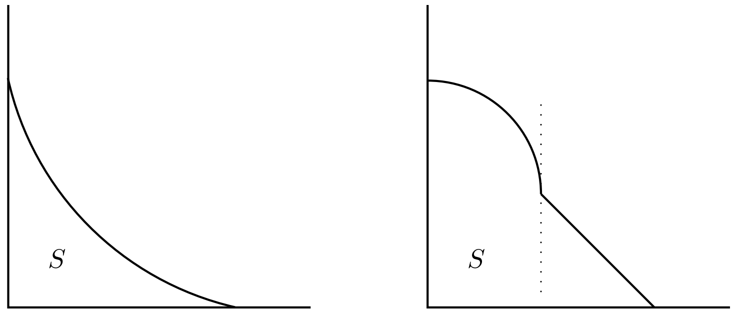
The interpretation of the bargaining problem is as follows. An outcome $x \in \mathbb{R}^n$ represents a utility allocation, in the sense that each x_i is the utility payoff obtained by player i; the *feasible set* S represents the set of all utility allocations players in N can jointly realize; players must find agreement on an outcome $x \in S$, and failure to do so leads to the unfavorable outcome $\mathbf{0}$. The point $\mathbf{0}$ is therefore also called the *disagreement point*.[5] The solution outcome $\varphi(S)$ is interpreted as the compromise reached by the players in N when faced with the problem S. Condition (A1) says that if an agent i gives up some of his utility $\varepsilon > 0$, then there is an upper bound $K\varepsilon$ on the associated compensation other agents (i.e., $j \in N \setminus \{i\}$) can feasibly attain.

Condition (A1) is a rather mild assumption. For instance, if the bargaining problem is a utility representation of simple economic division problem, then the condition already holds if the agents' utility functions are continuously differen-

[3] For $x, y \in \mathbb{R}^n$, $x \geq y$ is taken to mean $x_i \geq y_i$ for all $i \in N$; the vector inequalities $>$, $\leq$ and $<$ are similarly defined.

[4] For a closed set $S \in \mathbb{R}^n$, $\partial S := S \setminus \mathrm{int}(S)$, where $\mathrm{int}(S)$ denotes the interior of S.

[5] Normalization of the disagreement point to the zero vector $\mathbf{0}$ is without loss of generality.



(a) S satisfies (A1) (b) S violates (A1)

Figure 4.1. An illustration of condition (A1).

tiable. It is worth pointing out that Condition (A1) is not implied by convexity. Examples demonstrating this are easily constructed.

4.2.2 A Family of Raiffa Solutions

In order to define our solutions of interest, we first formalize the notion of a *maximal claims vector*. Given a bargaining problem $S \in \mathcal{B}$ and an outcome $x \in S \cap \mathbb{R}^n_+$, let $m(x, S) := (m_1(x, S), \ldots, m_n(x, S))$, where

$$m_i(x, S) := \max\{y_i | (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) \in S\}$$

for all $i \in N$. Each $m_i(x, S)$ is the maximal claim player i holds over the surplus that remains of S, given that an intermediate agreement has been reached on the outcome x. Note that $m(\cdot, S)$ is a well-defined vector function by strict comprehensiveness of S and compactness of the set $S \cap \mathbb{R}^n_+$. Whenever the problem S is understood, we write $m(x)$ rather than $m(x, S)$.

Given a convex problem $S \in \mathcal{B}$, the *discrete* Raiffa solution (Raiffa (1953)) is then defined as the limit of the sequence $\{x^t\}_{t=0}^{\infty}$, where $x^0 = \mathbf{0}$ and

$$x^{t+1} := x^t + \frac{1}{n}(m(x^t) - x^t) \tag{4.1}$$

for all $t \geq 1$. It is based on the intuitive notion that agreement is found on the midpoint of all the maximal claims agents hold over the surplus to divide. If this midpoint is not efficient, then agents again stake out their maximal claim over the surplus that remains, and reach a next compromise on the midpoint of those claims. The solution outcome is reached by iteratively applying this reasoning, until the entire surplus is allocated.

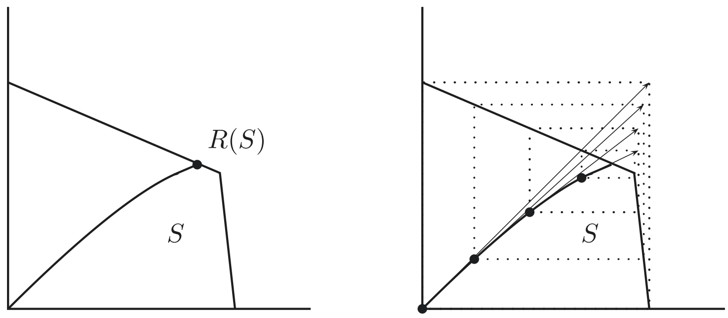


Figure 4.2. An illustration of the continuous Raiffa solution.

Note that if the problem S is not convex, then the midpoint of all maximal claims need not be feasible, and the discrete Raiffa solution may not be well-defined. This problem can be addressed by decreasing the step size $1/n$ in (4.1) to some $0 < c < 1/n$, as proposed by Diskin et al. (2011), or to 0, as proposed by Raiffa (1953). In the latter case, (4.1) becomes an initial value problem, and the sequence of intermediate agreements becomes an intermediate agreement *curve*. The limit point of this curve is our solution of interest, the *continuous Raiffa solution*. It has been considered for convex two-player problems by Raiffa (1953), Livne (1989), Peters and Damme (1991) among others; in these studies, the intermediate agreement curve is obtained as the solution of the initial value problem $dx_1/dx_2 = (m_1(x,S) - x_1)/(m_2(x,S) - x_2)$ with the initial condition $x_1(0) = 0$. In a multiple-player setting, the intermediate agreement curve could similarly be obtained by parameterizing the utilities of players $i \in N \setminus \{n\}$ in terms of the utility of player n. However, in their discussion of the continuous Raiffa solution, Diskin et al. (2011) explicitly used the continuous version of (4.1). Their definition can be generalized to a class of *weighted* Raiffa solutions.

Definition 23. *For* $\mu := (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n_{++}$, *the* continuous Raiffa solution $R^\mu : \mathcal{B} \to \mathbb{R}^n$ *is defined as*

$$R^\mu(S) := \lim_{t\to\infty} x(t)$$

where $x : [0, \infty) \to S$ is the unique solution of the Initial Value Problem[6]

$$x'(t) = \mu(m(x(t)) - x(t)) \quad \textit{and} \quad x(0) = \mathbf{0}. \tag{4.2}$$

The class $\mathcal{R} := \{R^\mu \mid \mu \in \mathbb{R}^n_{++}\}$ contains all such solutions.

Up to a constant $c > 0$, $R^{(c,\ldots,c)}$ is the unique symmetric solution in $\mathcal{R}$; this solution is also denoted R. An argument similar to Theorem 5 of Diskin et al. (2011) shows that all solutions in $\mathcal{R}$ are well-defined.

Proposition 4.2.1. *For all $S \in \mathcal{B}$ and $\mu \in \mathbb{R}^n_{++}$, problem (4.2) has a unique solution $x : [0, \infty) \to \mathbb{R}^n$, and $\lim_{t\to\infty} x(t)$ is contained in the boundary of S.*

All proofs are relegated to the Appendix.

4.3 A Non-Cooperative Bargaining Game

We consider a continuous-time bargaining game with stochastic timing of proposals and a finite deadline, similar to the game proposed by Ambrus and Lu (2015b). The underlying framework of this game is a bargaining problem $S \in \mathcal{B}$, as defined in the previous section. Bargaining occurs in a continuous time interval $[0, T]$, where the *deadline* T is finite and known to all players. For each player $i \in N$, the opportunity to make proposals is produced by a Poisson process with player-specific arrival rate $\lambda_i > 0$. These processes are assumed to be independent, and the associated arrival rates are further assumed to sum to one. The latter is without loss of generality, since the interval in which bargaining occurs can always be rescaled.

Whenever player i's process realizes, he proposes an allocation $x \in S$. Opponents then make instantaneous and sequential accept/reject decisions concerning this proposal. It is assumed that the order in which players decide on the made proposal corresponds with their indices – i.e. player k's decision precedes player l's decision if and only if $k < l$.[7] If all of i's opponents accept, then bargaining

[6] For $x, y \in \mathbb{R}^n$, we define $xy := (x_1y_1, \ldots, x_ny_n)$.

[7] The exact order in which players decide on made proposals is irrelevant.

ends with the implementation of the proposal. If at least one player rejects, then the game continues until the next arrival occurs and the above procedure is repeated. If no agreement is reached at or before the deadline T, then bargaining ends, and all players realize their disagreement value 0. A particular game of this form is described by $\Gamma = \{S, \lambda, T\}$, where $S \in \mathcal{B}$ is the underlying pure bargaining problem, $\lambda := (\lambda_1, \ldots, \lambda_n)$ the vector of players' arrival rates, and T the deadline ending negotiations.

4.3.1 Strategies

A strategy in a game $\Gamma = \{S, \lambda, T\}$ consists of two elements: which proposals to make when proposing, and which to accept or reject when responding. Which action a player chooses in either situation may depend on the history of play of the game. Consider a player $i \in N$. If he is the *proposer* at $t \in [0, T]$, then the history includes the times $0 \leq t_1 \leq \ldots \leq t_k < t$ of all previous offers (if any), and for each such time t_l, $l = 1, \ldots, k$, it specifies the corresponding proposal, the corresponding proposer, and the corresponding set of rejectors. If he is the *responder* at $t \in (0, T]$, then the history further includes the time-t proposal, the identity of its proposer, and the (possibly empty) set of rejectors so far.

Denote by H_i^p the set of all histories after which player i must make a proposal, and denote by H_i^r the set of all histories after which he must respond to a proposal. His *strategy* is then described by the pair (f_i, g_i), where $f_i : H_i^p \to S$ maps histories of H_i^p into feasible proposals $x \in S$, and $g_i : H_i^r \to \{Y, N\}$ maps histories of H_i^r into an accept/reject decision on the prevalent offer. A *strategy profile* is a tuple $(f, g) \equiv (f_i, g_i)_{i \in N}$.

4.3.2 Subgame Perfect Equilibrium

Heuristically, the construction of an SPE is based on the idea that proposers make offers such that responders are indifferent between accepting and rejecting. In particular, fixing a game $\Gamma = \{S, \lambda, T\}$, it is assumed that each agent $i \in N$ proposing at time $t \in [0, T]$, offers all opponents $j \in N \setminus \{i\}$ their respective reservation values – denoted $r_j(t)$ – and that he claims $p_i(t)$ for himself. Furthermore, it is assumed that such proposals are accepted.

These assumptions allow us to derive an expression for agents' reservation values. Consider again a player i in N. At any time $t \in [0, T]$ the next realization of any of the n concurrent processes occurs at a time s in the interval $[t, T]$. The probability that it is player i's process that then realizes, is given by λ_i; with probability $1 - \lambda_i$ some other process realizes first. Thus, player i's expected utility payoff at time s is $u_i(s) = \lambda_i p_i(s) + (1 - \lambda_i) r_i(s)$. Since the waiting time until the first next offer is exponentially distributed with rate 1, we obtain the following expression for $r_i(t)$:

$$r_i(t) = \int_t^T e^{-(s-t)} [\lambda_i p_i(s) + (1 - \lambda_i) r_i(s)] ds.$$

After offering all agents $j \neq i$ their reservation values $r_j(t)$, a proposer i claims the utility that makes his proposed allocation efficient. That is, $p_i(t) = m_i(r(t))$. This leads to the following system of equations:

$$r_i(t) = \int_t^T e^{-(s-t)} [\lambda_i p_i(s) + (1 - \lambda_i) r_i(s)] ds, \tag{4.3a}$$

$$p_i(t) = m_i(r(t)). \tag{4.3b}$$

for all $i \in N$ and $t \in [0, T]$. It turns out that it has a unique solution.

Lemma 4.3.1. *System* (4.3) *has a unique solution* $(p^*, r^*) : [0, T] \to \mathbb{R}^n \times \mathbb{R}^n$.

Similar to Rubinstein (1982), a strategy profile can be constructed based on the solution to system (4.3).

Definition 24. (f^*, g^*) *is a strategy profile, such that for all* $i \in N$ *and for all* $t \in [0, T]$*:*

- *If i is proposing at time t, he offers $r_j^*(t)$ to all $j \neq i$, and claims $p_i^*(t)$ for himself.*
- *If i is responding at time t, he accepts a proposal v iff $v_i \geq r_i^*(t)$.* *(4)*

An argument similar to Claim 3 of Ambrus and Lu (2015b) shows that (f^*, g^*) is the unique SPE of the game.

Proposition 4.3.2. *(f^*, g^*) is the unique SPE of the game Γ.*

4.4 Main Result

This section investigates the behavior of the payoffs (p^*, r^*) associated with the unique SPE (f^*, g^*) as the horizon T tends to infinity. Consider the game Γ from the previous section. Figure 4.3 shows the SPE payoffs of a player $i \in N$, as a function of the time $t \in [0, T]$.

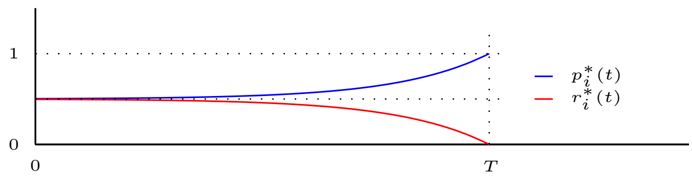


Figure 4.3. The SPE payoffs of a player $i \in N$.

The shape of these payoff curves is intuitive: the closer players get to the horizon T, the higher the probability the game will run out without another offer being made, and thus the higher the cost of rejecting a proposal. This means that, as the game approaches T, responders will have to accept lower offers, and proposers can make higher claims for themselves. This reasoning underlines the history-independent nature of the SPE: which proposals players make, or agree on, depends only on the time that remains until the game expires. It is thus useful to define functions x and y that specify players' SPE payoffs as a function of the remaining time to T. More specifically, $x(t) := r^*(T - t)$ and $y(t) := p^*(T - t)$. It is then sufficient to study the limit behavior of the functions x and y. To see this, suppose that after the start of the game, the first process realizes at some time $\bar{t} \in [0, T]$. Then the game concludes at $\bar{t}$ and the payoffs players realize are given by $x(T - \bar{t})$ and $y(T - \bar{t})$. Since the first arrival is exponentially distributed with a unit rate parameter, $\bar{t}$ is finite, meaning $(T - \bar{t})$ will tend to infinity with T.

The functions x and y are derived from system (4.3). In particular, by (4.3b)

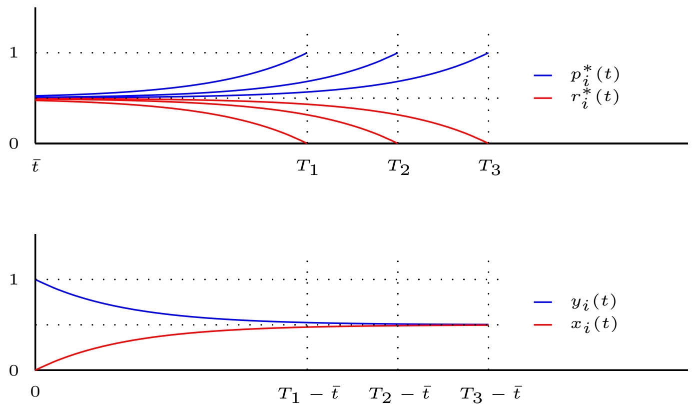


Figure 4.4. Time-$\bar{t}$ SPE payoffs with deadlines $T_3 > T_2 > T_1$.

$y(t) = m(x(t)) = m(r^*(T-t)) = p^*(T-t)$, and by (4.3a),

$$\begin{aligned}
x_i(t) &= r_i^*(T-t) \\
&= \int_{T-t}^{T} e^{-(s-T+t)}[\lambda_i p_i^*(s) + (1-\lambda_i) r_i^*(s)]ds \\
&= \int_0^t e^{(\tau-t)}[\lambda_i p_i^*(T-\tau) + (1-\lambda_i) r_i^*(T-\tau)]d\tau \\
&= \int_0^t e^{(\tau-t)}[\lambda_i y_i(\tau) + (1-\lambda_i) x_i(\tau)]d\tau.
\end{aligned}$$

for all $i \in N$. In the first place, this implies $x(0) = \mathbf{0}$. Furthermore, differentiating with respect to t yields

$$\frac{dx_i(t)}{dt} = \lambda_i(m_i(x(t)) - x_i(t)) \tag{4.5}$$

Thus, we obtain (4.2), the Initial Value Problem that defines the λ-weighted Raiffa solution, where $\lambda = (\lambda_1, \ldots, \lambda_n)$. It then follows from the definition that $x(T-\bar{t})$ converges to $R^\lambda(S)$ as T diverges. Since $y(t) = m(x(t))$ for all t, it follows from continuity of $m(\cdot)$ that $y(T-\bar{t})$ converges to $m(R^\lambda(S))$; since $R^\lambda(S) \in \partial S$, it follows that $y(T-\bar{t})$ converges to $R^\lambda(S)$ as well. Hence, also the payoff of the time-$\bar{t}$ proposer converges to the value implied by the λ-weighted Raiffa solution. We may summarize as follows.

Theorem 4.4.1. *The payoffs associated with the unique SPE of a game* $\Gamma = \{S, \lambda, T\}$ *converge to* $R^{\lambda}(S)$ *as the horizon* T *tends to infinity.*

Remark. There are two potential criticisms on Theorem 4.4.1. In the first place, it only provides *approximate* non-cooperative support for the Raiffa solution. More seriously, for every finite horizon, there is a strictly positive probability that bargaining ends before any player has the opportunity to make a proposal. In such a case players realize zero payoffs, without an action ever being played. However, using the approach of Trockel (2011), both criticisms may be tackled at once. In particular, consider an extension of the game Γ in which the deadline T is not exogenously specified, but rather, it is *chosen* by the first rejector of the first proposal. Then the game does not conclude before an offer is made, the first proposer proposes exactly the (weighted) Raiffa solution, and all opponents immediately accept. Hence, it yields an *exact* support result for R^{μ}.

Remark. Theorem 4.4.1 should not be confused with Theorem 2 of Ambrus and Lu (2015b). They consider a coalitional bargaining framework, where the underlying cooperative game is a convex TU game, while we take the underlying game to be a pure bargaining problem. Of course, for TU games where generating any surplus requires the grand coalition, Ambrus and Lu (2015b) also implement the continuous Raiffa solution, but the restriction to TU games means they only do so on the domain of bargaining problems for which the Pareto set is a linear transformation of the $(n-1)$-dimensional unit simplex. In such bargaining problems, the λ-weighted continuous Raiffa solution coincides with the λ-weighted discrete Raiffa solution, the λ-weighted Nash bargaining solution, the λ-weighted Kalai-Smorodinsky solution, and many others. The distinction arises in pure bargaining problems where utility is not transferable, indeed, the framework considered in this paper. This does not mean that this set-up is more general; as mentioned before, TU games and pure bargaining problems are simply different subsets of the class of NTU games.

4.5 An Alternative Proposer Protocol

Selten (1981) studied an elegant alternative proposer protocol in which the player who rejects the current proposal is called to make the next proposal. This protocol - also named the *rejector-proposes* protocol - has been studied primarily in the context of coalitional bargaining games, and has been shown to have potentially important implications for the resulting equilibria (see e.g. Chatterjee and K. (1993)). In this section it is demonstrated that this is not the case in the present game. That is, under the rejector-proposes protocol, the SPE is unchanged, SPE *payoffs* are unchanged, and these payoffs continue to converge to the Raiffa solution as the deadline of negotiations tends to infinity. This is in line with the findings of Britz et al. (2010), Britz and Predtetchinski (2012), who consider a bargaining game that provides non-cooperative support for the asymmetric Nash bargaining solution; whether the underlying protocol is action-independent or whether the designated next proposer is the last rejector, turns out to be immaterial to their support result.[8]

The Game As before, bargaining occurs in continuous time, in an interval that ranges from 0 to T with $T > 0$, the rate at which a player $i \in N$ can make proposals is governed by a Poisson process with player-specific arrival rate λ_i, and without loss of generality it is assumed that $\sum_i \lambda_i = 1$.

The main difference with respect to the game defined above is that players' processes no longer run concurrently. Instead, there is always a single designated next proposer who will make his offer at the first next arrival of his own Poisson process. It is assumed that player $\hat{i} \in N$ is the designated next proposer at time 0. A second departure from the previous game is that the proposer also votes on his own proposal, and moreover, that he is the first to do so. In particular, we assume that if player i puts an offer on the table, then the order of votes is given by $[i, i+1, \ldots, n, 1, \ldots, i-1]$ if $i > 1$, and by $[1, \ldots, n]$ otherwise.[9] Unanimous agreement on a proposal continues to end the game with the implementation of that proposal. However, if unanimous agreement is not reached, then the game continues, but now with the first rejector in the above-defined order in the role of designated next proposer. As before, if no unanimous agreement is reached before

[8] Britz and Predtetchinski (2012) in fact study a more general action-dependent proposer protocol that includes the rejector-proposes protocol as a special case.

[9] While the exact order of the players deciding after the proposer continues to be irrelevant, it is essential that the proposer decides on his own offer first.

or at time T, then players realize the disagreement outcome $\mathbf{0}$. Such a game is denoted by $\Gamma^{\text{RP}} = \{S, \lambda, N\}$, where S, λ and T are as defined above.

Strategies In a game Γ^{RP}, strategies are somewhat simpler to define. In particular, rather than specifying the identities of all previous proposers, histories only need to specify who is the first proposer. The identities of all subsequent proposers can then be inferred from the play of the game. Thus, a history in H_i^p specifies the first designated next proposer $\hat{i} \in N$, and further specifies for each $t \in (0, T]$ the times $0 \leq t_1 \leq \ldots \leq t_k < t$ of all previous proposals (if any), the corresponding proposals, and the corresponding sets of rejectors. A history in H_i^r additionally specifies the time-t proposal, and the set of rejectors prior to i. A strategy for player i is again a pair of functions (f_i, g_i) with $f_i : H_i^p \to S$ and $g_i : H_i^r \to \{Y, N\}$; a strategy profile is again denoted by (f, g).

Subgame Perfect Equilibrium A heuristic reasoning is helpful in the construction of an SPE. Consider a game Γ^{RP}, and assume that players play a strategy profile (f, g), such that proposers make all opponents indifferent between accepting and rejecting, and furthermore that such proposals are immediately accepted. Suppose a player i's associated time-t (expected) payoff, given that player $j \in N$ is called to be the next proposer, is denoted $q_i^j(t)$. Since the first rejector is called to be the next proposer, and since the proposal prescribed by f is assumed to be accepted by all, deviating leads to the payoff $r_i(t) := q_i^i(t)$. Hence, if player i is the proposer at time t, he offers $r_j(t)$ to all opponents $j \neq i$. Denoting the payoff i realizes himself as $p_i(t)$, this yields

$$r_i(t) = \int_t^T \lambda_i e^{-\lambda_i(s-t)} p_i(s) ds \tag{4.6a}$$

$$p_i(t) = m_i(r(t)) \tag{4.6b}$$

for all $i = 1, \ldots, n$ and $t \in [0, T]$. Observe that $r_i(T) = 0$. Furthermore, differentiating equation (4.6a) yields $r_i'(t) = -\lambda_i p_i(t) + \lambda_i r_i(t)$. Thus, the solutions to systems (4.3) and (4.6) coincide.

Proposition 4.5.1. *System* (4.6) *has a unique solution* $(\hat{p}, \hat{r}) : [0, T] \to \mathbb{R}^n \times \mathbb{R}^n$. *Moreover,* $(\hat{p}, \hat{r}) = (p^*, r^*)$.

This allows for defining a strategy profile $(\hat{f}, \hat{g})$ that is analogous to (f^*, g^*). In particular, given a time-t history in H_i^p, a proposer i's strategy $\hat{f}_i$ is to propose $(p_i^*(t), r_{-i}^*(t))$; given a time-t history in H_i^r that includes the time-t proposal $v \in S$, a responder i's strategy $\hat{g}_i$ is to accept v if and only if $v_i \geq r_i^*(t)$. An argument analogous to Proposition 4.3.2 then demonstrates that the change of protocol has no influence on the outcome of the game.

Proposition 4.5.2. *$(\hat{f}, \hat{g})$ is the unique SPE in the game Γ^{RP}.*

4.6 Concluding Remarks

This paper has provided a non-cooperative foundation for the continuous Raiffa solution in multilateral bargaining problems. Moreover we showed that this foundation does not rely on the convexity of the bargaining set. While the game introduced to this end is rather natural, it does include the somewhat unrealistic assumption that players do not discount their payoffs over time. A natural extension would thus be to allow for the discounting of utilities. In this case, a connection seems to arise with the Nash bargaining solution. It may in the first place be conjectured that the game enriched with a discount factor continues to have a unique SPE in stationary strategies, and that the associated SPE payoffs continue to converge to a stationary point as the deadline T tends to infinity. Given that the feasible set satisfies a certain smoothness condition, this stationary point will in fact converge to the Nash bargaining solution as the discount factor tends to one (see e.g. Kultti and Vartiainen (2010)). Exploring the modified version of the game with a discount factor in more detail is left for future work.

.1 Proofs

.1.1 Proof of Proposition 4.2.1

Consider $S \in \mathcal{B}$, and let $\|\cdot\|$ denote the taxicab metric – i.e., for $x \in \mathbb{R}^n$, $\|x\| := \sum_{i=1}^n |x_i|$. Then we have the following useful lemma.

Lemma .1.1. *The function $m(\cdot)$ is Lipschitz continuous on $D := \{z \in S \mid z' \leq z \leq z''$ where $z', z'' \in S \cap \mathbb{R}^n_+$ with $z' < z''\}$, with Lipschitz constant $L = nK$.*

Proof. Consider $v, w \in D$, and define the sequence $\{z^k\}_{k=0}^n$ where $z^0 := v$, and for all $k = 1, \ldots, n$, $z^k := (w_1, \ldots, w_k, v_{k+1}, \ldots, v_n)$. Then each z^k is an element of D, and z^k and z^{k-1} only differ in the k-th coordinate. For $k \in N$ and $i \in N \setminus \{k\}$, we have that the points $(m_i(z^k), z^k_{-i})$ and $(m_i(z^{k-1}), z^{k-1}_{-i})$ are in $\partial S \cap \mathbb{R}^n_+$. Since $S \in \mathcal{B}$, it then follows from condition (A1) that

$$|m_i(z^k) - m_i(z^{k-1})| \leq K|z^k_k - z^{k-1}_k| = K|w_k - v_k|.$$

In addition, $|m_k(z^k) - m_k(z^{k-1})| = 0 \leq K|w_k - v_k|$. Then for all $i \in N$ we have

$$\begin{aligned} |m_i(w) - m_i(v)| &= \left| \sum_{k=1}^n m_i(z^k) - m_i(z^{k-1}) \right| \leq \sum_{k=1}^n |m_i(z^k) - m_i(z^{k-1})| \\ &\leq \sum_{k=1}^n K|w_k - v_k| = K\|w - v\|. \end{aligned}$$

Therefore, $\|m(w) - m(v)\| = \sum_{i=1}^n |m_i(w) - m_i(v)| \leq nK\|w - v\|$. □

Fix some $\mu \in \mathbb{R}^n_{++}$ and define $f(x) := \mu(m(x) - x)$. By Lemma .1.1, the function f satisfies a uniform Lipschitz condition on $S \cap \mathbb{R}^n_+$. Then by the Picard-Lindelöf theorem it follows that problem (4.2) has a unique solution $x(t)$. Consider the maximal interval of existence $[0, \omega)$ of this solution.

(i) For all $z \in \text{int}(S) \cap \mathbb{R}^n_+$, $f(z) > \mathbf{0}$. Hence, $x(t)$ is a strictly increasing function. Then by the extension theorem (e.g. Theorem 8.33 of Kelley and Peterson (2010)), it follows that $x(t)$ converges to a point $\hat{x} \in \partial S \cap \mathbb{R}^n_+$ as $t \to \omega$.

(ii) Assume that ω is finite. Then the function $v(t) := x(\omega - t)$ is the unique solution to the problem $[v'(t) = -f(v)$ and $v(0) = \hat{x}]$. Since $v'(0) = \mathbf{0}$, it follows that $v(t) = \hat{x}$ for all $t \in [0, \omega]$. Since this implies $\mathbf{0} = x(0) = v(\omega) = \hat{x}$ – a contradiction – it follows that $\omega = \infty$. □

.1.2 Proof of Lemma 4.3.1

The first equation of the obtained system (4.3) can be equivalently written as a differential equation. In particular, for $i \in N$ we have

$$\begin{aligned}\frac{dr_i(t)}{dt} &= 0 - \frac{dt}{dt}e^{-(t-t)}[\lambda_i p_i(t) + (1-\lambda_i)r_i(t)] \\ &\qquad + \int_t^T e^{-(s-t)}[\lambda_i p_i(s) + (1-\lambda_i)r_i(s)]ds \\ &= -[\lambda_i p_i(t) + (1-\lambda_i)r_i(t)] + r_i(t) \\ &= -\lambda_i m_i(r(t), S) + \lambda_i r_i(t)\end{aligned}$$

Furthermore, $r(T) = \bar{0}$. Then the proof follows along the lines of Proposition 4.2.1. □

.1.3 Proof of Proposition 4.3.2

Consider the game Γ, and for $t \in [0, T]$, let $\overline{r}_i(t)$ and $\underline{r}_i(t)$ respectively be player i's associated supremum and infimum reservation values over all SPE's, and all time-t histories in H_i^r. Assume that

$$\underline{r}(t) = \overline{r}(t) = r^*(t) \quad \text{and} \quad r^*(t) \in \text{int}(S) \quad \text{for all } t \in [\hat{T}, T], \tag{7}$$

where $\hat{T}$ is some time in $(0, T]$, r^* represents players' reservation values under strategy profile (f^*, g^*), and $\text{int}(S)$ again denotes the interior of S. The aim of the proof is to show that (7) also holds on a non-trivial time interval that precedes $\hat{T}$. The following lemma helps define that interval.

Lemma .1.2. *There exists a $c > 0$ such that for all $x \in int(S) \cap \mathbb{R}^n_+$, we have that $x + c(m(x) - x) \in S$.*

Proof. Let $x \in \text{int}(S) \cap \mathbb{R}^n_+$, let $y := x + \alpha^*(m(x) - x)$ where $\alpha^* := \max\{\alpha \mid x + \alpha(m(x) - x) \in S\}$, and let $F := \{z \in \mathbb{R}^n \mid x \leq z \leq y\}$. By Lemma .1.1 it follows that for all $v, w \in F$, we have

$$\|(m(w) - w) - (m(v) - v)\| \leq (1 + nK)\|w - v\|. \tag{8}$$

Note that $m(y) = y$, so $\|(m(x) - x) - (m(y) - y)\| = \|m(x) - x\|$. Furthermore, $\|x - y\| = \|x - x - \alpha^*(m(x) - x)\| = \alpha^*\|m(x) - x\|$. Then by (8), it follows that

$$\|m(x) - x\| \leq \alpha^*(1 + nK)\|m(x) - x\|.$$

Since $x \in \text{int}(S) \cap \mathbb{R}^n_+$, this implies $\alpha^* \geq \frac{1}{1+nK}$. It follows that for all $x \in \text{int}(S)$, $x + \frac{1}{1+nK}(m(x) - x) \in S$. □

Let $\tau := \ln \frac{2}{2+c}$, where c is as in Lemma .1.2. Then the probability of another arrival in the interval $[\hat{T} - \tau, \hat{T}]$ is given by $c/2$

Lemma .1.3. *For all $t \in [\hat{T} - \tau, \hat{T}]$ we have $\overline{r}(t) \in int(S)$.*

Proof. Let v be an SPE proposal accepted at a time $t \in [\hat{T} - \tau, \hat{T}]$. Any player $i \in N$ can secure the payoff $r_i^*(\hat{T})$ by

(i) rejecting the proposal v at time t and all subsequent proposals at times $t' \in (t, \hat{T}]$, and

(ii) claiming $r_i^*(\hat{T})$ at any time $t' \in [t, \hat{T}]$ where he himself is the proposer.

Then SPE implies that $v \geq r^*(\hat{T})$. Assume next that there is a player $i \in N$ with $v_i > m_i(r^*(\hat{T}))$. Since then there is a player $j \neq i$ for whom $v_j < r_j^*(\hat{T})$, contradicting the above, it follows that $v \leq m(r^*(\hat{T}))$. It follows that expected payoffs in SPE are bounded between $r^*(\hat{T})$ and $m(r^*(\hat{T}))$.

Consider a player i and a time $t \in [\hat{T} - \tau, \hat{T}]$. If no more arrivals occur within the interval $[t, \hat{T}]$, player i realizes the payoff $r_i^*(\hat{T})$; if on the other hand a process *does* realize within this interval, then i's payoff is bounded above by $m_i(r^*(\hat{T}))$. The former occurs with a probability $e^{-(\hat{T}-t)}$, and the latter with probability $(1 - e^{-(\hat{T}-t)})$. Hence,

$$\begin{aligned}
\overline{r}_i(t) &\leq e^{-(\hat{T}-t)} r_i^*(\hat{T}) + (1 - e^{-(\hat{T}-t)}) m_i(r^*(\hat{T})) \\
&= r_i^*(\hat{T}) + (1 - e^{-(\hat{T}-t)})(m_i(r^*(\hat{T})) - r_i^*(\hat{T})) \\
&\leq r_i^*(\hat{T}) + \frac{c}{2}(m_i(r^*(\hat{T})) - r_i^*(\hat{T})) \\
&< r_i^*(\hat{T}) + c(m_i(r^*(\hat{T})) - r_i^*(\hat{T}))
\end{aligned}$$

The lemma then follows by Lemma .1.2 and comprehensiveness of S. □

Lemma .1.4. *For all $t \in [\hat{T} - \tau, \hat{T}]$ and $i \in N$ we have*

$$\underline{r}_i(t) = \int_t^{\hat{T}} e^{-(s-t)}[\lambda_i m_i(\overline{r}(s)) + (1-\lambda_i)\underline{r}_i(s)]ds + r_i^*(\hat{T}) \quad \textit{and} \tag{9a}$$

$$\overline{r}_i(t) = \int_t^{\hat{T}} e^{-(s-t)}[\lambda_i m_i(\underline{r}(s)) + (1-\lambda_i)\overline{r}_i(s)]ds + r_i^*(\hat{T}). \tag{9b}$$

Proof. Consider a player i, and define the following strategy profile. For all $t \in [0, T]$:

- Player i offers $\overline{r}_j(t)$ to all $j \neq i$, and claims $m_i(\overline{r}(t))$ himself. Players $j \neq i$ accept an offer v iff $v_j \geq \overline{r}_j(t)$. In case of disagreement, the game moves to an SPE in which the first rejector j receives $\underline{r}_j(t)$.

- Player $k \neq i$ offers $\underline{r}_i(t)$ to player i, $\overline{r}_j(t)$ to all $j \neq i, k$, and claims $m_k(\underline{r}_i(t), \overline{r}_{-i}(t))$ for himself. Player i accepts a proposal v iff $v_i \geq \underline{r}_i(t)$, player $j \neq i, k$ accepts v iff $v_j \geq \overline{r}_j(t)$. In case k's proposal is rejected, then the game moves on to an SPE in which first rejector j realizes the payoff $\underline{r}_j(t)$.

It follows from Lemma .1.3 that for $t \in [\hat{T} - \tau, \hat{T}]$, the above proposals are feasible. It is further immediate that responders in this strategy profile cannot profitably deviate from their strategies. To see that the same is true for proposers, consider first the case where i is proposing. By Lemma .1.3 and the definition of $m(\cdot)$, it follows that $\overline{r}(t) < m(\overline{r}(t))$, and thus $\overline{r}_i(t) < m_i(\overline{r}(t))$; it follows that player i has no profitable deviation. Consider a proposer $k \neq i$, and observe that $\overline{r}_k(t) < m_k(\overline{r}(t))$ by similar reasoning as before. Moreover, $m_k(\overline{r}(t)) \leq m_k(\overline{r}_i(t), \underline{r}_i(t))$. If he deviates from the above strategy, then k's expected payoff is dominated by $\overline{r}_k(t)$, and thus by $m_k(\overline{r}_i(t), \underline{r}_i(t))$. Hence, player k cannot profitably deviate either.

It follows that for all $t \in [\hat{T} - \tau, \hat{T}]$, we have

$$\begin{aligned}
\underline{r}_i(t) &= \int_t^T e^{-(s-t)}[\lambda_i m_i(\overline{r}(s)) + (1-\lambda_i)\underline{r}_i(s)]ds \\
&= \int_t^{\hat{T}} e^{-(s-t)}[\lambda_i m_i(\overline{r}(s)) + (1-\lambda_i)\underline{r}_i(s)]ds \\
&\qquad + \int_{\hat{T}}^T e^{-(s-t)}[\lambda_i m_i(r^*(s)) + (1-\lambda_i)r_i^*(s)]ds \\
&= \int_t^{\hat{T}} e^{-(s-t)}[\lambda_i m_i(\overline{r}(s)) + (1-\lambda_i)\underline{r}_i(s)]ds + r_i^*(\hat{T}).
\end{aligned}$$

All other cases are similar. □

From Lemma 4.3.1 it follows that $(\overline{r}(t), \underline{r}(t)) = (r^*(t), r^*(t))$ is the *unique* solution to system (9). Hence, condition (7) holds on the interval $[\hat{T} - \tau, T]$.

Observe that $\underline{r}(T) = \overline{r}(T) = \bar{0} = r^*(T)$ and $\mathbf{0} \in \text{int}(S)$. Thus, since $\tau > 0$ and T is finite, iteratively applying the above argument leads in a finite number of steps to the conclusion that $\underline{r}(t) = \overline{r}(t) = r^*(t)$ for all $t \in [0, T]$. It then follows that the strategy profile (f^*, g^*) is the unique SPE of the game Γ. □

.1.4 Proof of Proposition 4.5.2

The argument is similar to Proposition 4.3.2. Attention is focused on the steps that differ.

- Let $\overline{r}_i(t)$ and $\underline{r}_i(t)$ be player i's supremum respectively infimum expected payoff over all SPE's, and all time-t histories in H_i^r that are such that, player i is the designated next proposer. Again, assume that for all $t \in [\hat{T}, T]$, we have $\underline{r}(t) = \overline{r}(t) = r^*(t)$, and that $r^*(t)$ is contained in int(S).

- Consider $c > 0$ as defined in Lemma .1.2, but now define $\tau := \ln\left(\frac{2}{2+c}\right)^{\frac{1}{\tilde{\lambda}}}$ where $\tilde{\lambda} := \max_{i \in N} \lambda_i$. Then whichever player is called to be the next proposer, the probability of another arrival in the interval $[\hat{T} - \tau, \hat{T}]$ is at most $c/2$.

- Consider an SPE in which a proposal v is accepted at some time $t \in [\hat{T} - \tau, \hat{T}]$. Suppose that $v_i > m_i(r^*(\hat{T}))$ for some player $i \in N$. Then there is a player

$j \in N \setminus i$ for whom $v_j < r^*_j(\hat{T})$. As before, such a player can profitably deviate from his strategy. In particular, he can realize $r^*_j(\hat{T})$ by rejecting v at time t, becoming the designated next proposer, and rejecting each offer he subsequently gets to make in the interval $(t, \hat{T}]$. This again implies that any *expected* SPE payoff in the interval $[\hat{T} - \tau, \hat{T}]$ is below $m(r^*(\hat{T}))$.

Suppose that player i is called to be the next proposer at a time $t \in [\hat{T} - \tau, \hat{T}]$. The probability of his process realizing within the interval $[t, \hat{T}]$ is $(1 - e^{-\lambda_i(\hat{T}-t)})$, and the probability that no arrival occurs is $e^{-\lambda_i(\hat{T}-t)}$. In the former case he realizes a payoff of at most $m_i(r^*(\hat{T}))$, in the latter case he realizes a payoff in excess of $r^*_i(\hat{T})$. By the same reasoning as in Proposition 4.3.2 we then obtain $\overline{r}_i(t) < r^*_i(\hat{T}) + c(m_i(r^*(\hat{T})) - r^*_i(\hat{T}))$, which by Lemma .1.2 and comprehensiveness implies $\overline{r}(t) \in \mathrm{int}(S)$.

- Analogous to Proposition 4.3.2 we can define $2n$ strategy profiles, feasible and optimal in the interval $[\hat{T} - \tau, \hat{T}]$, such that

$$\underline{r}_i(t) = \int_t^{\hat{T}} \lambda_i e^{-\lambda_i(s-t)} m_i(\overline{r}(s))ds + r^*_i(\hat{T}) \quad \text{and} \tag{10a}$$
$$\overline{r}_i(t) = \int_t^{\hat{T}} \lambda_i e^{-\lambda_i(s-t)} m_i(\underline{r}(s))ds + r^*_i(\hat{T}). \tag{10b}$$

for all $i \in N$. The unique solution of this system on the interval $[\hat{T} - \tau, \hat{T}]$ is again $(\underline{r}(t), \overline{r}(t)) = (r^*(t), r^*(t))$. As before, a finite number of iterations of this argument leads to the conclusion that $r^*(t)$ is the unique solution on the entire interval $[0, T]$, and thus that the strategy profile $(\hat{f}, \hat{g})$ is the unique SPE of the game. □

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Universidad Carlos III de Madrid

Essays in Microeconomic Theory

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Thesis submitted to the Department of Economics of the Universidad Carlos III de Madrid for the degree of Doctor of Philosophy, Getafe, February 2016.

Declaration

I certify that the thesis I have presented for examination for the PhD degree of the Department of Economics at the Universidad Carlos III de Madrid is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

I warrant that this authorization does not, to the best of my belief, infringe the rights of any third party.

Abstract

My thesis considers various aspects of microeconomic theory and focuses on the different types of uncertainty that players can encounter. Each chapter studies a setting with a different type of uncertainty and draws conclusions about how players are likely to behave in such a situation.

The first chapter focuses on games of incomplete information and is joint work with Nora Wegner. We provide conditions to allow modelling situations of asymmetric information in a tractable manner. In addition we show a novel relationship between certain games of asymmetric information and corresponding games of symmetric information. This framework establishes links between certain games separately studied in the literature. The class of games considered is defined by scalable preference relations and a scalable information structure. We show that this framework can be used to solve asymmetric contests and auctions with loss aversion.

In the second chapter I move to situations in which information is almost complete. In joint work with Nora Wegner, we consider the robustness of subgame perfect implementation in situations when the preferences of players are almost perfectly known. More precisely we consider a class of information perturbations where in each state of the world players know their own preferences with certainty and receive almost perfectly informative signals about the preferences of other players. We show that implementations using two-stage sequential move mechanisms are always robust under this class of restricted perturbations, while those using more stages are often not.

The third chapter deals with a case of complete information and is joint work with

Nora Wegner. We introduce the family of weighted Raiffa solutions. An individual solution is characterised by two parameters representing the bargaining weight of each player and the speed at which agreement is reached. First we provide a cooperative foundation for this family of solutions, by appealing to two of the original axioms used by Nash and a simple monotonicity axiom. Using similar axioms we give a new axiomatization for a family of weighted Kalai-Smorodinsky solutions. Secondly we provide a non-cooperative foundation for weighted Raiffa solutions, showing how they can be implemented using simple bargaining models where offers are intermittent or the identity of the proposer is persistent. This shows that weighted Raiffa solutions have cooperative foundations closely related to those of the Kalai-Smorodinksy solution, and non-cooperative foundations closely related to those of the Nash solution.

The fourth chapter is closely related to the third chapter and is joint work with Bram Driesen and Nora Wegner. It provides a non-cooperative foundation for asymmetric generalizations of the continuous Raiffa solution. Specifically, we consider a continuous-time variation of the classic Stahl-Rubinstein bargaining model, in which each player's opportunity to make proposals is produced by an independent Poisson process, and a finite deadline ends the negotiations. Under the assumption that future payoffs are not discounted, it is shown that the payoffs realized in the unique subgame perfect equilibrium of this game approach the continuous Raiffa solution as the time horizon tends to infinity. The weights reflecting the asymmetries among the players, correspond with the Poisson arrival rates of their respective proposal processes

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Chapter 1

Scalable Games: Modelling Games of Incomplete information

1.1 Introduction

It is commonly known that there are many economic situations, where each party knows something which the other parties do not know. For instance, a company may know its cost of producing a certain product but not know the cost of its competitors. Alternatively in a common value auction a bidder may know how much he thinks the object is worth but not know the estimates of other bidders. In our analysis we will refer to these situations where each party has some private information as *games of asymmetric information.*

The main contribution of this paper is to introduce an information structure that ensures situations of asymmetric information can be modelled in a tractable manner.

The key condition we consider is on the nature of players' private information - the scalable information structure. This condition ensures that a player's private information - his signal - does not provide him with information about how his signal compares to that of other players. That is to say after observing his signal a player cannot infer anything about whether the signal observed is a relatively high or relatively low compared to those of his opponents. We say that the scalable information structure exhibits *maximal rank uncertainty*, because a player's private information does not provide him with information about the rank of his payoff type.

To illustrate this information structure, consider the following examples. Abreu and Brunnermeier (2003) study the formation of asset price bubbles. Investors learn about the existence of a bubble, but they cannot infer whether they are likely to be among the first or the last to learn about the existence of a bubble. Similarly in the double auctions studied by Satterthwaite et al. (2014), bidders and the seller do not know whether their valuation is likely to be higher or lower than that of the other bidders - and the seller. Their valuation does not provide them with information about the rank of their signal. The same information structure is also used in an application to contests and auctions with loss averse players considered in this paper.

All of the above are examples where a scalable information structure has been used to model a situation of asymmetric information which is intractable under alternative modelling assumptions. The equilibria in these examples are simple and easy to find. We refer to them as *constant strategy equilibria* as they are described by a single parameter for each player. In Abreu and Brunnermeier (2003)'s model of the formation of asset price bubbles, once he learns about the existence of the bubble, each investor decides to ride the bubble for a fixed amount of time independent of the time at which he learns. In the double auction example, all bidders shade their valuation by a constant amount, whatever their valuation. In our application to auctions and contests with loss averse bidders, players bid a constant proportion of their valuation or the effort exerted is a constant proportion of their ability respectively. The framework suggested here therefore allows us to study and make revenue comparisons for such auctions, which have received considerable attention in the literature, but cannot be solved under alternative modelling assumptions.[1]

Our entire analysis is restricted to games where players' preferences satisfy a mild

[1]See Gill and Prowse (2012), Lange and Ratan (2010) and Eisenhuth and Ewers (2015) for example.

condition, which is naturally satisfied in many situations and includes all cases where preferences are homogeneous of some degree k among others.

While the focus of our analysis lies on situations of asymmetric information, the theoretical contribution of our paper is to provide a link between games of asymmetric information and games of symmetric information.

One can think of a common situation where all parties share the same information, but there is some additional information, which cannot be accessed by any of the parties. Extreme weather events are one example of such a situation, since all parties have the same information from a central weather forecast agency but still face uncertainty. In our analysis, these situations where all parties involved have the same information but nevertheless there is some uncertainty are referred to as *games of symmetric information.*

We show that under the scalable information structure and mild restrictions on players' preferences, games of asymmetric information are strategically equivalent to games of symmetric information. Although these games differ only in what is observed by the players, they are typically used to study very different situations. Our framework therefore provides a novel link between seemingly unrelated games. The relevance of this link is illustrated by demonstrating a link between a second price auction with a reserve price where in one case participants know their valuation, but do not know the reserve price and in the other case they know the reserve price but are uncertain about their valuation. Another application using our framework to solve an otherwise complex situation considers the bargaining process to dissolve a business following bankruptcy.

The remainder of the paper is structured as follows. In the remainder of this section we relate the suggested approach to the literature. In section two, we illustrate the use of the framework in two simple examples. The key property of maximal rank uncertainty is discussed in detail section three. Section four introduces the model. The scalable structure is presented in section five, while section six contains the simplicity analysis. In section seven we present present an application to auctions with loss aversion to illustrate the simplicity and tractability of our model.

Restricting attention to settings where players' preferences can be represented by utility functions, we introduce the equivalence between scalable games of asymmetric information and scalable games of symmetric information in section eight. The relevance of this link is illustrated in section nine. Section ten concludes.

1.1.1 Related Literature

In the literature, models which satisfy the scalable preference and scalable information structure considered in this paper have been used to model specific situations of uncertainty. As mentioned above, the formation of asset price bubbles studied by Abreu and Brunnermeier (2003) and the double auctions considered by Satterthwaite et al. (2014) are two such models. Other examples include the clock games considered by Brunnermeier and Morgan (2010), as well as supply function competition studied by Vives (2011). While these papers provide models for specific situations, we aim at providing a general tool to model situations of asymmetric information.

The information structure of the proposed class of games of asymmetric information, referred to as scalable games has close links with the literature on *global games* introduced by Carlsson and Van Damme (1993) and considered in Morris and Shin (2002) and Morris and Shin (2003) among others. As in global games, players face uncertainty about the state of the world θ which is drawn from a diffuse prior. Moreover each player does not observe θ but instead receives a partially informative signal s_i about the state of the world, where $s_i = \theta + z_i$ and z_i can be interpreted as a noise term. However, in global games the main objective is equilibrium selection which arises since coordination is more difficult when the state of the world is unknown. Moreover the games considered in our paper do not necessarily have dominance regions and a player's signal typically enters his payoff function directly. Above all the focus of this paper lies on the characterization of equilibria in games of asymmetric information rather than equilibrium selection in games of complete information.

The framework proposed in our paper also has close ties with the literature on *quadratic utility models* [2] In these games there is also uncertainty about the state of the world and players receive a noisy signal of the state. Quadratic utility models typically focus on the social value of information and the role of information acquisition.[3] Applications to Cournot competition are provided by Vives (1988) and Myatt and Wallace (2013).

As in our paper, each player receives a signal about the state of the world which can be interpreted as his type and may enter his payoff function directly. The payoff function in most quadratic utility models depend on the actions of others only through the aggregate. Scalable games of asymmetric information require weaker conditions on the preference structure - for example allowing for loss aversion - at the cost of making stronger distributional assumptions on the state and the signals: the information structure in a quadratic utility model is affine, satisfying the assumption that $E[\theta|s_i] = \alpha s_i + \beta$; in the related scalable game in additive form we require the shape of the distribution to be the same for all types and hence $E[\theta|s_i] = s_i + \beta$.

In a recent paper, Morris et al. (2015) propose the concept of uniform rank belief. When there are two players, the authors say that players have a uniform rank belief if each of them assigns probability $\frac{1}{2}$ to having a higher payoff type than his opponent independent of his payoff type. Meanwhile the maximal rank uncertainty property suggested in this paper, says that the probability each player assigns to being in any particular rank is independent of his type, but it need not be equal to $\frac{1}{2}$ or $\frac{1}{n}$ in the case of n players.

Finally considering a translation from one game to a strategically equivalent game, which is easier to solve, has been proposed by Baye and Hoppe (2003) in the case of rent seeking and patent races. However they consider relationships between games of complete information, while we consider translations from a game of asymmetric information to a game of symmetric information. The aim to model situations of

[2] A comprehensive treatment of these games is provided in Angeletos and Pavan (2007) for a continuum of players, while Ui and Yoshizawa (2014) consider a discrete number of players.

[3] For models with endogenous information structures see for example Colombo and Pavan (2014) Myatt and Wallace (2012) and Pavan (2014).

incomplete information in a tractable manner is also pursued by Compte and Postlewaite (2013) who consider a private value first price auction, where bidders shade their bid by a constant amount, independent of their valuation.

1.2 Illustrative Examples

We now introduce a simple example to illustrate the strategic equivalence of certain games that are closely related, but have a different information structure. Three cases are distinguished (i) a game of asymmetric information, where a player faces uncertainty about the signals of other players, (ii) the case where players have symmetric information, but nevertheless there is some uncertainty and (iii) a complete information game.

1.2.1 Single-player Example

Consider a game, where there is one buyer wanting to buy a product. His valuation for the product is given by $s \in (0, \infty)$. The reserve price for the product is given by $\theta \in (0, \infty)$. The buyer offers to pay a fraction of his valuation $a \in \{\frac{1}{3}, \frac{1}{2}\}$. Hence, the suggested price is given by $p(s) = as$. In case the price offered by the player is higher than the reserve price, he obtains a payoff of $u(a, s, \theta) = s(1 - a)$ if$sa \geq \theta$. If the proposed price is below the reserve price the buyer obtains a payoff of zero: $u(a, s, \theta) = 0$ if $\theta > as$.

Suppose that θ is determined according to an improper prior with density function $g(\theta) = \frac{1}{\theta}$ for all $\theta \in (0, \infty)$. For any level of θ, the conditional distribution of s is given as follows:

$$f(s|\theta) = \begin{cases} \frac{1}{4} & \text{if } s = 2\theta \\ \frac{3}{4} & \text{if } s = 3\theta \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

First consider the game where the buyer observes his valuation s but does not know the reserve price θ. This is the game we refer to as *scalable game of asymmetric information*. By Bayesian updating the buyer assigns a probability of $g(\theta|s) = \frac{2}{11}$

to the case $\theta = \frac{s}{2}$ and assigns the remaining probability $g(\theta|s) = \frac{9}{11}$ to the case $\theta = \frac{s}{3}$.

The buyer's expected payoff from offering $a = \frac{1}{2}$ is given by $E\Big[u(\frac{1}{2}, s, \theta)|s\Big] = s(1 - \frac{1}{2}) = \frac{s}{2}$. Meanwhile his expected payoff from choosing $a = \frac{1}{3}$ is given by $E\Big[u(\frac{1}{3}, s, \theta)|s\Big] = \frac{9}{11}s(1 - \frac{1}{3}) = \frac{6s}{11}$. Hence, the buyer prefers to offer $a = \frac{1}{3}$ independent of his valuation.

Now consider instead the case where the buyer observes the reserve price - the price displayed at a shop - but does not know his valuation. This is the game we refer to as *scalable game of symmetric information.*

His expected payoff from offering $a = \frac{1}{2}$ is given by $E\Big[u(\frac{1}{2}, s, \theta)|\theta\Big] = \frac{1}{4}2\theta(1 - \frac{1}{2}) + \frac{3}{4}3\theta(1 - \frac{1}{2}) = \frac{11\theta}{8}$. Meanwhile his expected payoff from choosing $a = \frac{1}{3}$ is given by $E\Big[u(\frac{1}{3}, s, \theta)|\theta)\Big] = \frac{3}{4}3\theta(1 - \frac{1}{3}) = \frac{3\theta}{2}$. Again the buyer prefers to offer $a = \frac{1}{3}$ independent of the reserve price.

Moreover, note that the ratio of choosing $a = \frac{1}{3}$ to choosing $a = \frac{1}{2}$ is the same in both cases and is given by $\frac{12}{11}$. As will become clear later, this structure satisfies our scalability assumptions. We will show that these two games are equivalent.

Secondly, these games were very easy to solve. The optimal decision of a player does not depend on his valuation or the reserve price respectively. In fact any games in the class of scalable games proposed in this paper can be solved by looking at the optimal action for a buyer with a valuation $s = 1$ (or a reserve price $\theta = 1$). It is not necessary to consider the optimal decision for each valuation (reserve price) separately. This also means that the game is strategically equivalent to a game of complete information which one could choose to solve instead.

1.2.2 Multi-player Example

To further illustrate this concept and show that the framework can also capture games with several players and using a different information structure, we now present a second example.

Consider a world with two competing countries labeled $\{1, 2\}$ who actively exert their influence in a certain region. At time θ a new militant group emerges, which threatens the security of one country but furthers the interests of the other.

Each country does not immediately learn of this new development, but rather finds out at some time s_i. After learning of existence of the militant group, each country must choose how long to wait until deciding upon a response. This waiting time is denoted by $a_i \geq 0$. It is assumed that decisions are immediately put into action. Since better intelligence will lead to more effective intervention, it is assumed that the payoff associated with executing an action after waiting for a time a_i is $\overline{u}_i(a_i, s_i) = a_i$. However so that the two countries do not enter into direct conflict, only the first action chosen is executed, and the second mover receives a payoff of $\underline{u}_i(a_i, s_i) = 0$. It is assumed that countries have no prior information about when the new group will emerge, and this is modelled by θ being drawn from a diffuse prior with $g(\theta) = 1$ for all $\theta \in \mathbb{R}$. Furthermore we assume that $s_i = \theta + z_i$, where each z_i is independent of θ and is distributed uniformly over the interval $[0, 1]$. Each country observes its signal, the time at which it learns s_i, but does not observe θ.

As will be clear from the formal definition that follows this game is a scalable game of asymmetric information.

In order to solve this scalable game we look for a symmetric equilibrium in constant strategies of the form $\sigma_i(s_i) = a^*$. Since constant strategies directly imply monotonicity of reactions, the maximization problem of country i can be written as follows:

$$\max_{a_i} \int_{s_i - 1}^{s_i} a_i g(\tilde{\theta}|s_i)(1 - F(s_i + a_i - a_j|\tilde{\theta}))d\tilde{\theta}$$

The key condition we consider is on the nature of players' private information - the scalable information structure. This condition ensures that a player's private information does not provide him with information about how his payoff type compares to that of other players. That is to say after observing his signal a player cannot infer anything about whether the signal observed is a relatively high or relatively low compared to those of his opponents. The scalable information structure exhibits *maximal rank uncertainty*, because a player's private information does not provide him with information about the rank of his payoff type.

$$\int_{s_i-1}^{s_i}(1-F(s_i+a_i-a_j|\tilde{\theta}))d\tilde{\theta}=\int_{s_i-1}^{s_i}f(s_i+a_i-a_j|\tilde{\theta})d\tilde{\theta}$$

Since we are looking for a symmetric equilibrium $a_i = a_j$. Moreover from the information structure, we know that $\int_{s_i-1}^{s_i}(1-F(s_i|\tilde{\theta}))d\tilde{\theta} = 0.5$ for all s_i. Independent of its signal, each country is always equally likely to have the lower or to have the higher signal. We also know that $\int_{s_i-1}^{s_i} f(s_i|\tilde{\theta})d\tilde{\theta} = 1$ and hence $\sigma_i(s_i) = 0.5$. It turns out that this strategy is an equilibrium in constant strategies.

Consider now that instead of delay both countries learn of the emergence of the new militant group immediately and hence observe the state θ. Again each country chooses how long to wait until deciding upon its response, a_i. However in this version of the game there is a delay between the decision to act and the implementation of the action itself. This delay is given by $z_i = s_i - \theta$, where again z_i is drawn from a uniform distribution over the interval $[0, 1]$ for each $i \in \{1, 2\}$. Country i is the first mover only if $a_i + s_i < a_j + s_j$ and in this case country i receives a payoff of $\overline{u}_i(a_i, s_i) = a_i$. The second mover again receives a payoff of $\underline{u}(a_i, s_i) = 0$. The maximisation problem for each country looks as follows:

$$\max_{a_i} a_i\Big(1-\int_{\theta}^{\theta+1}F_i(\tilde{s}_i+a_i-a_j|\theta)d\tilde{s}_i\Big)$$

Taking first order conditions leads to:

$$1-\int_{\theta}^{\theta+1}F_i(\tilde{s}_i+a_i-a_j|\theta)d\tilde{s}_i=a_i\int_{\theta}^{\theta+1}f(\tilde{s}_i+a_i-a_j|\theta)$$

In a symmetric equilibrium we know that $a_i = a_j$. Moreover by the scalable information structure, $\int_{\theta}^{\theta+1} F_i(\tilde{s}_i + a_i - a_j|\theta))d\tilde{s}_i = 0.5$ and $\int_{\theta}^{\theta+1} f(\tilde{s}_i|\theta)d\tilde{s}_i = 1$. Therefore $\sigma_i = 0.5$ for i, j is an equilibrium of this game.

1.3 Key property

The key property driving the results in these examples is what we refer to as *maximal rank uncertainty.* We now consider this property in some more detail. Suppose there is a set of players $I = \{1, 2, ..., n\}$. Further suppose that each player i receives a signal $s_i \in \mathbb{R}$.

We now define the rank r_i of player i as follows. Let the set $\overline{I}_i = \{j : s_j \geq s_i\}$, so that $j \in \overline{I}_i$ only if the signal of player j is greater or equal to the signal of player i. With this notation in mind, define $r_i = |\overline{I}_i|$. This means that r_i captures the number of players with a signal greater than or equal to s_i. Hence if $r_i = 1$ then player i has the highest signal amongst all players and if $r_i = n$ then player i has the lowest signal amongst all players. The key property of our model can be informally stated as follows:

$$P(r_i = m|s_i) = P(r_i = m|s_i') \text{ for all } s_i, s_i', m, i$$

This equation captures that fact that the probability any player i assigns to having the n-highest signal is independent of his signal. It means that a player's signal does not give him information about his relative position compared to other players. This contrasts with a model where players have independent types. For instance consider a two player model where (i) $I = \{1, 2\}$, (ii) s_i and s_j are drawn from a uniform distribution over $[0, 1]$ and (iii) s_i and s_j are drawn independently. In this case:

$$P(r_i = 1|s_i) = s_i \text{ for all } s_i$$

This equation captures the fact that a player who observes a signal s_i close to 1 is very confident that he has the highest signal and $s_i = 1$, while a player who observes a signal s_i close to 0 is very confident that he has the lower signal and $r_i = 2$.

Hence in a model with independent types players gain *rank information* about their relative position compared to other players when they observe their signal. Meanwhile the key property in our model ensures that players do not gain *rank* *information* from observing their signal. We believe this to be a more appropriate way to model certain situations such as some auctions where a bidder's valuation may not help him decide whether he has the highest valuation or not (this would be the case in situations where having a higher valuation increases the likelihood that other bidders also have a high valuation).

1.4 The Model

We now introduce the general model and formally define a class of games with maximal rank uncertainty, capturing the illustrative examples above.

Consider a finite set of players $I = \{1, ..., n\}$. The state is denoted by $\theta \in \Theta = (\underline{\theta}, \overline{\theta})$ and each player is associated with a signal $s_i \in S_i = (\underline{s}_i, \overline{s}_i)$. In most applications this signal can be thought of as a player's type and hence describing his preferences. For simplicity we consider $S_i = \Theta$ for all $i \in I$. We expect the results to hold for any open interval S_i.

Each player i simultaneously and independently chooses an action $a_i \in A_i \subseteq \mathbb{R}$. Action sets may be player specific. To ease notation we use $\mathbf{s} = (s_1, \ldots, s_n)$ to denote the vector of players' signals and $\mathbf{a} = (a_1, \ldots, a_n)$ to denote the vector of players' actions. Moreover we define $\omega = (\mathbf{a}, \theta, \mathbf{s})$ to be an outcome described by a vector of actions $\mathbf{a}$ the state θ and the vector of all players signals $\mathbf{s}$. Let Ω denote the set of outcomes.

In order to cover both expected and certain non-expected utility frameworks, we state players' preference relations in terms of lotteries over outcomes. A lottery $L \in \mathbb{L}$ is a cumulative distribution over the outcomes Ω, $L : \Omega \mapsto [0, 1]$, where an outcome ω is given by $(\mathbf{a}, \theta, \mathbf{s})$. Each player has a preference relation over the set of lotteries $\mathbb{L}$. It is assumed that these preference relations are complete, continuous and transitive, but crucially we do not assume the independence axiom.

The information structure is given as follows. The distribution of each player's signal s_i is given by $F_i(s_i^*|\theta)$. These distributions are assumed to be independent conditional on θ and $F(\mathbf{s}|\theta)$ denotes the distribution of all players' signals conditional on θ. In addition we assume that these conditional distributions have a density, which we denote by $f_i(s_i|\theta)$.

It is crucial in our analysis that we allow θ to have an improper prior distribution. To this aim, we define the prior over θ with a function $g : \Theta \to [0, \infty)$. The probability that $\theta \leq \theta^*$ given s_i, $G(\theta^*|s_i)$ is given as follows:

$$G_i(\theta^*|s_i) = \int_{\underline{\theta}}^{\theta^*} \frac{f_i(s_i|\theta)g(\theta)}{\int_\Theta f_i(s_i|\tilde{\theta})g(\tilde{\theta})d\tilde{\theta}} d\theta. \tag{1.2}$$

The case of a proper prior corresponds to the case in which (i) g plays the role of a density and (ii) (1.2) is the standard Bayes rule. However, the above formulation also allows for improper priors in which $\int_\Theta g(\theta)d\theta = \infty$. We use $g_i(\theta^*|s_i)$ to denote the probability density function corresponding to $G_i(\theta^*|s_i)$.

To simplify notation, we use Γ to summarise the primitives of the model:

$$\Gamma \equiv \{I, \Theta, (A_i)_{i\in I}, (\succeq_i)_{i\in I}, g, (F_i)_{i\in I}\}$$

Two cases of this basic model are studied in our paper. The difference lies in the source of the uncertainty. First we consider the case where each player privately observes his signal s_i, but does not observe the state. This is the game we refer to as a *game of asymmetric information*. Second we consider the case where all players observe the state, but do not observe their private signals s_i. This is the game we refer to as a *game of symmetric information*.

We write $\mathcal{A}(\Gamma)$ to denote the game of asymmetric information where each player privately observes s_i while θ is not observed. Similarly $\mathcal{S}(\Gamma)$ is used to denote the game of symmetric information where $\theta$is commonly known among all players, but the signals s_i are not observed.

1.4.1 Strategies

In order to avoid introducing additional notation, we will jointly define the strategies used in games of asymmetric information and games of symmetric information, despite the differences in what is observed by the players.

A strategy for player i is described by a cumulative distribution function over actions, conditional on the state θ and on the player's signal s_i and is denoted by $\sigma_i(a_i, \theta, s_i)$. This notation allows us to capture both mixed strategies and pure strategies succinctly. We use σ to denote $(\sigma_1, \ldots, \sigma_n)$.

In a game of asymmetric information $\mathcal{A}(\Gamma)$, players do not observe the state θ and hence feasible strategies are constant in θ. The set of strategies which are constant in θ is denoted by $\Sigma^{\mathcal{A}}$, as it is the set of feasible strategies under asymmetric information: $\sigma_i^{\mathcal{A}}(a_i, \theta, s_i) = \sigma_i^{\mathcal{A}}(a_i, \theta', s_i)$ for all θ, θ', s_i and a_i. A typical element in this set for player i is denoted by $\sigma_i^{\mathcal{A}}$.

Similarly, in a game of symmetric information $\mathcal{S}(\Gamma)$, players do not observe their signal s_i, and we require the strategy to be constant in s_i. The set of such strategies is denoted by $\Sigma^{\mathcal{S}}$ and describes the feasible strategies in a game of symmetric information: $\sigma_i^{\mathcal{S}}(a_i, \theta, s_i) = \sigma_i^{\mathcal{S}}(a_i, \theta, s_i')$ for all θ, s_i, s_i' and a_i. A typical element in this set for player i is denoted by $\sigma_i^{\mathcal{S}}$.

Our analysis makes use of strategies which are constant in both s_i and θ. We refer to these strategies as *constant strategies*. The set of these strategies is given by Σ^C and a typical element in this set for player i is denoted σ_i^C: $\sigma_i^C(a_i, \theta, s_i) = \sigma_i^{\mathcal{A}}(a_i, \theta', s_i')$ for all $\theta, \theta', s_i, s_i'$ and a_i. A typical element in this set for player i is denoted by $\sigma_i^{\mathcal{A}}$.

In the examples mentioned in the introduction, these constant strategies correspond to all investors riding the bubble for a fixed amount of time, shading their valuation by a constant amount in the double auction or bidding a constant fraction of their valuation respectively.

1.4.2 Equilibria

Suppose (i) player i has observed a signal s_i^*, (ii) player i chooses an action $a_i \in A_i$ and (iii) other players play according to the strategy profile $\sigma^{\mathcal{A}}$ where $\sigma^{\mathcal{A}}(\mathbf{a}|\mathbf{s}) = \prod_{j \in I} \sigma^c(a_j)$. We define $L_i^{\mathcal{A}}[s_i^*; a_i, (\sigma_j^c)_{j \neq i}]$ to capture the weights that player i assigns to different outcomes in this situation. Hence:

$$L_i^{\mathcal{A}}[s_i^*; a_i, (\sigma_j^c)_{j\neq i}](\mathbf{a}, \mathbf{s}, \theta) = \begin{cases} \left(\prod_{i\neq j} \sigma_j^c(a_j) \right) \int_{\underline{\theta}}^{\theta} \int_{\underline{\mathbf{s}}_{-\mathbf{i}}}^{\mathbf{s}_{-i}} g(\tilde{\theta}|s_i^*) f(\tilde{\mathbf{s}}_{-i}|\tilde{\theta}) d\tilde{\mathbf{s}}_{-i} d\tilde{\theta} \text{ if } s_i \geq s_i^* \text{ and } a_i \geq a_i^* \\ 0 \text{ otherwise} \end{cases}$$

The equilibrium for a game of asymmetric information $\mathcal{A}(\Gamma)$ can now be defined as follows:

Definition 1 (Constant strategy equilibrium: Game of asymmetric information)**.** *A strategy profile $\sigma^{\mathcal{A}} \in \Sigma^{\mathcal{A}}$ is a constant strategy equilibrium of the game $\mathcal{A}(\Gamma)$ if for all players $i \in I$ and for all signals $s_i^* \in S_i$ (i) $\sigma^{\mathcal{A}}(\mathbf{a}|\mathbf{s}^*) = \prod_{j\in I} \sigma_j^c(a_j)$ and (ii) for all actions $a_i^* \in supp(\sigma_i^c)$ and all deviations $\hat{a}_i \in A_i$ it holds that:*

$$L_i^{\mathcal{A}}[s_i^*; a_i^*, (\sigma_j^c)_{j\neq i}] \succeq_i L_i^{\mathcal{A}}[s_i^*; \hat{a}_i, (\sigma_j^c)_{j\neq i}]$$

This definition says that the constant strategy profile $\sigma^{\mathcal{A}}$ is an equilibrium, if each player i - given that he observes signal s_i^* - weakly prefers the lottery generated by choosing choosing any optimal action $a_i^* \in \text{ supp}\sigma_i^c$ compared to the lottery generated by choosing any alternative action $\hat{a}_i$. Although this definition only considers constant strategy profiles, it allows for arbitrary deviations. Hence a constant strategy equilibrium of $\mathcal{A}(\Gamma)$ is also a Bayesian Nash equilibrium of $\mathcal{A}(\Gamma)$.

Suppose now that (i) the state is known to be θ^*, (ii) player i chooses action a_i^* and (iii) other players play according to the strategy profile $\sigma^S(\mathbf{a}|\theta^*)$ where $\sigma^S(\mathbf{a}|\theta^*) = \prod_{j\in I} \sigma_j^c(a_j)$. We define $L_i^{\mathcal{S}}[\theta^*; a_i^*, (\sigma_j^c)_{j\neq i}]$ to capture the weights that player i assigns to different outcomes in this situation. Therefore:

$$L_i^{\mathcal{S}}[\theta^*; a_i, (\sigma_j^c)_{j\neq i}](\mathbf{a}, \mathbf{s}, \theta) = \begin{cases} \left(\prod_{i\neq j} \sigma_j^c(a_j) \right) \int_{\underline{\mathbf{s}}}^{\mathbf{s}} f(\tilde{\mathbf{s}}|\theta^*) d\tilde{\mathbf{s}} \text{ if } \theta \geq \theta^* \text{ and } a_i \geq a_i^* \\ 0 \text{ otherwise} \end{cases}$$

A constant strategy equilibrium in this game of symmetric information $\mathcal{S}(\Gamma)$, can now be defined as follows:

Definition 2 (Constant strategy equilibrium: Game of symmetric information)**.** *A strategy profile $\sigma^{\mathcal{S}} \in \Sigma^{\mathcal{S}}$ is a constant strategy equilibrium of the game $\mathcal{S}(\Gamma)$ if* *for all states* $\theta^* \in \Theta$ *and for all players* $i \in I$ *(i)* $\sigma^{\mathcal{S}}(\mathbf{a}|\theta^*) = \prod_{j\in I} \sigma_j^c(a_j)$ *and (ii) for all actions* $a_i^* \in supp(\sigma_i)$ *and all deviations* $\hat{a}_i \in A_i$ *it holds that:*

$$L_i^{\mathcal{S}}[\theta^*; a_i^*, (\sigma_j^c)_{j\neq i}] \succeq_i L_i^{\mathcal{S}}[\theta^*; \hat{a}_i, (\sigma_j^c)_{j\neq i}]$$

This definition says that the constant strategy profile $\sigma^{\mathcal{S}}$ is an equilibrium, if each player i - given that the state is θ - weakly prefers the lottery generated by choosing choosing any optimal action $a_i^* \in$ suppσ_i compared to the lottery generated by choosing any alternative action $\hat{a}_i$. Again note that while this definition only considers constant strategy profiles, it allows for arbitrary deviations and hence constant strategy equilibria of $\mathcal{S}(\Gamma)$ are also Bayesian Nash equilibria of $\mathcal{S}(\Gamma)$. In the next section we propose conditions on the preference relation and information structure which help ensure that constant strategy equilibria exist.

1.5 Scalable primitives

We now propose conditions on the players' preference relations and the information structure which lead to games with the desired scalability properties.

In order to define scalable games in a general framework, we use a generator and an operator to state the required conditions. A *generator*, denoted by H, is a strictly increasing bijection from Θ to $\mathbb{R}$. We also assume that it is differentiable.[4] Secondly the *operator* associated to the generator H, denoted by $\oplus_H$, maps any two numbers $(a, b) \in \Theta^2$ into the unique number $a \oplus_H b \in \Theta$ that solves:

$$a \oplus_H b \equiv H^{-1}\Big(H(a) + H(b)\Big)$$

The operator $\ominus_H$ is defined symmetrically as $a \ominus_H b \equiv H^{-1}\Big(H(a) - H(b)\Big)$. [5] An obvious example is $H(x) = x$. In this case, the operators $\oplus_H$ and $\ominus_H$ are the usual sum and subtraction, respectively. Another example is the case when

[4] The assumption of differentiability is made in order to simplify calculations. We believe that this assumption is not necessary.

[5] The terms $\oplus_H$ and $\ominus_H$ can be thought of the the normal $+$ and $-$ after a projection of the state space from $\mathbb{R}$ to Θ.

Table 1.1. The generator

Θ	$H(\theta)$	$a \oplus_H b$	$a \ominus_H b$
$\mathbb{R}$	θ	$a+b$	$a-b$
$\mathbb{R}_{++}$	$\frac{1}{\theta}$	$a \times b$	$a \div b$

$H(x) = \ln(x)$. Here the operators $\oplus_H$ and $\ominus_H$ are the usual multiplication and division, respectively.
In some cases it is useful to consider a reference point for either the signal of player i or the state. For a generator H, we use $0_H := x$ such that $H(x) = 0$.[6] Returning to the illustrative examples, the reference point in the case of a single buyer wanting to buy an object would be his valuation $s = 1$, or the reserve price $\theta = 1$, while in the two country example, the reference point corresponds to the case where a firm learns about the existence of the military group at time zero or where the military group is formed at time zero.

1.5.1 Scalable preference relations

Given an outcome $\omega = (\mathbf{a}, \theta, \mathbf{s})$,let $\omega \oplus_H k \equiv (\mathbf{a}, \theta \oplus_H k, \mathbf{s} \oplus_H k)$ and let $[L \oplus_H k](w) \equiv L(w \oplus_H k)$. This allows us to introduce scalable preference relations.

Definition 3 (Scalable preference relations)**.** *A preference relation $\succeq_i$ is scalable with respect to H if whenever:*

$$L_i \succeq_i L_i'$$

then,

$$[L_i \oplus_H k] \succeq_i [L_i' \oplus_H k]$$

This definition says that if player i prefers lottery L_i to the lottery L_i', then he will also prefer the lottery corresponding to scaling up the signals of all players and the state by some constant k using the notion of scalability given by H and keeping all actions constant, in lottery L, to a similarly scaled version of the lottery L'. This preference structure is naturally satisfied in many situations. All

[6]The choice of this reference point is arbitrary and has no particular meaning.

examples mentioned in the introduction - including auctions and contests with loss averse players - exhibit scalable preference relations. Moreover this general preference structure allows us to capture and hence study situations which cannot be modelled using expected utility, such as the auctions with loss averse players. However this general structure is not always necessary. We will later present a simple sufficient condition for it to be satisfied.

1.5.2 Scalable information structure

The second key element of our analysis is the information structure. A scalable information structure is defined as follows:

Definition 4 (Scalable information structure)**.** *The information structure* $\{g, \{F_i(s_i|\theta)\}_{i\in I}\}$ *is scalable with respect to* H *if:*

1. $g(\theta) = H'(\theta)$ *for all* $\theta \in \Theta$
2. *For all* $\theta, k, s_i \in \Theta$
$$F_i(s_i|\theta) = F_i(s_i \oplus_H k|\theta \oplus_H k)$$

The first part of this definition ensures that the notion of scalability used in the information structure corresponds is appropriate for the primitives.

Part two of this definition captures the fact that the conditional distribution of signals has a similar shape when θ is changed. When $a \oplus_H b = a + b$ this implies that conditional beliefs are additively invariant: that is to say the shape of the distribution is common knowledge but players do not know their position in the distribution. For instance this holds when players know that the distribution is uniform over the interval $[\theta - 1, \theta + 1]$, but do not necessarily know the value of the state θ. This is illustrated in Figure 1.1.

Meanwhile when $a \oplus_H b = a \times b$ this definition implies that conditional beliefs are homogeneous of degree 0. For instance this holds when players know that the distribution is uniform over the interval $[0, 2\theta]$, but do not necessarily know the value of the median θ. This is illustrated in Figure 1.2.

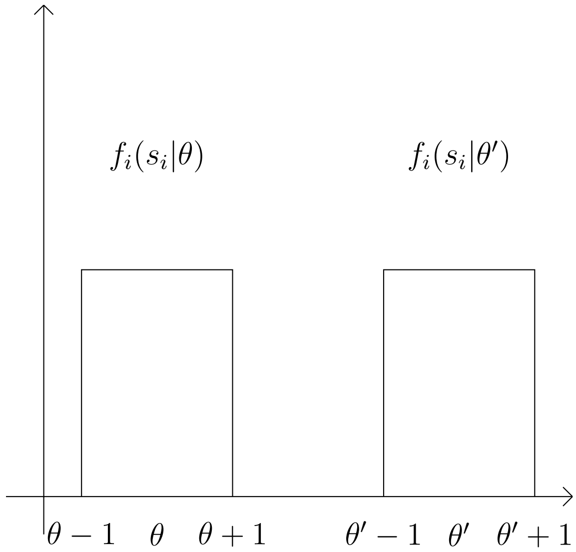


Figure 1.1. Uniform:Additive

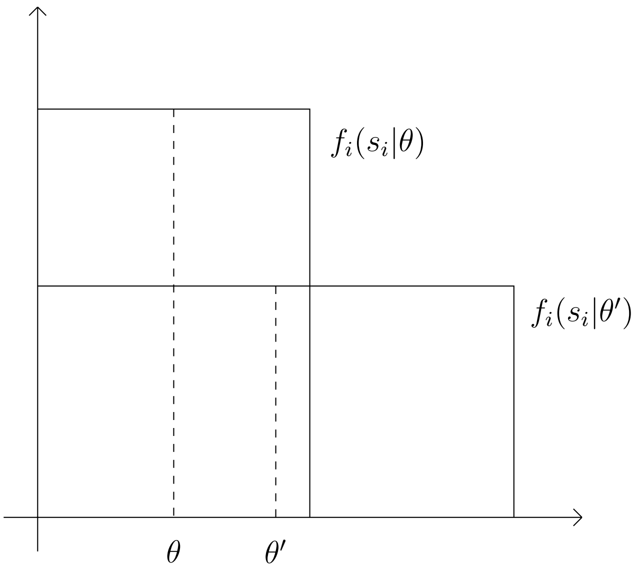


Figure 1.2. Uniform: Multiplicative

1.5.3 Scalable Games

Considering a structure with primitives given by Γ, where players simultaneously choose an action, we say that the structure is *scalable* if the preference relations are scalable (see definition 3.5.1) and the information structure is scalable (see definition 4).

More precisely, combining the notion of a scalable information structure with the definition of $\mathcal{S}(\Gamma)$ and $\mathcal{A}(\Gamma)$, we define the following:

Definition 5 (Scalable game of asymmetric information)**.** *We say that the game of asymmetric information $\mathcal{A}(\Gamma)$ is a scalable game of asymmetric information, if the preferences $(\succeq_i)_{i\in I}$ are scalable (see definition 3.5.1) and the information structure $\{g,(F_i)_{i\in I}\}$ is scalable (see definition 4).*

Definition 6 (Scalable game of symmetric information)**.** *We say that the game of symmetric information $\mathcal{S}(\Gamma)$ is a scalable game of symmetric information, if the preferences $(\succeq_i)_{i\in I}$ are scalable (see definition 3.5.1) and the information structure $\{g,(F_i)_{i\in I}\}$ is scalable (see definition 4).*

These are the two types of games to which we apply our framework.

1.6 Analysis: Simplicity

In this section we show that scalable games are particularly tractable. This is demonstrated by drawing the connection between scalable games of asymmetric information $\mathcal{A}(\Gamma)$ and an associated game of complete information.

Informally, scalable games are particularly easy to solve, because in order to determine the optimal strategy for each player, it is sufficient to look at one particular signal or state for each player. The optimal actions are the same, when he has a different signal, or the state is different. In the case of pure strategies, the problem is reduced from solving for a fixed point in the space of functions to solving for a fixed point in the space of vectors, one for each player.

This simplicity can also be demonstrated by considering a related game of complete information.

Let $L_i^{\mathcal{A}}[\sigma^c, 0_H]$ represent the lottery that player i assigns to possible outcomes when (i) player i observes signal $s_i = 0_H$ and (ii) players play according to constant strategy profile $\sigma(\mathbf{a}|\mathbf{s}) = \sigma^c(\mathbf{a})$. As a reminder:

$$L_i^{\mathcal{A}}[\sigma^c, 0_H](\mathbf{a}, s_i, s_{-i}, \theta) = \begin{cases} \sigma^c(\mathbf{a}) \int_\theta \int_{s_{-i}} g(\theta|0_H) f(s_{-i}|\theta) ds_{-i} d\theta \text{ when } s_i \geq 0_H \\ 0 \text{ otherwise} \end{cases}$$

Using this notation we can now define the complete information game $\mathcal{C}(\Gamma)$:

Definition 7 (Complete information game $\mathcal{C}(\Gamma)$)**.** *The complete information game corresponding to the primitives* Γ *is given by* $\mathcal{C}(\Gamma) := \{I, (A_i)_{i \in I}, (\succeq_i^c)_{i \in I}\}$ *where:*

$$\sigma^c \succeq_i^c \hat{\sigma}^c \text{ if and only if } L_i^{\mathcal{A}}{}_{[\sigma^c]} \succeq_i L_i^{\mathcal{A}}{}_{[\hat{\sigma}^c]}$$

This leads us to the following result:

Proposition 1.6.1 (Game of complete information)**.** *For goven primitives* Γ*, the constant strategy profile* σ^c *is a Bayesian Nash equilbrium of the scalable game* $\mathcal{A}(\Gamma)$*, if and only if it is a Nash equilibrium of the complete information game* $\mathcal{C}(\Gamma)$*.*

Hence a scalable game where players have a symmetric information - denoted by $\mathcal{A}(\Gamma)$ - is particularly easy to solve because it is sufficient to study a corresponding game of complete information $\mathcal{C}(\Gamma)$. If we also require that (i) the independence axiom holds (so that preferences can be represented by a utility function) and (ii) each action space A_i is finite then it is a standard result that the complete information game $\mathcal{C}(\Gamma)$ must have an equilibrium (possibly in mixed strategies). Then by appealing to proposition 1.6.1, we can ensure that an equilibrium in constant strategies also exists in the game of asymmetric information $\mathcal{A}(\Gamma)$. Even when these conditions do not hold, an equilibrium can often be found by studying the game of complete information $\mathcal{C}(\Gamma)$.

Being able to focus on equilibria in constant strategies, makes scalable games easier to solve than more general games of incomplete information. When looking for an equilibrium in constant strategies, the problem reduces from looking for a fixed point in the space of functions to finding a fixed point in the space of vectors.

1.7 Applications: Simplicity

To show that the simplicity of the scalable game framework allows to study situations which are difficult to model under alternative assumptions, we now present an application to auctions and contests with loss averse participants.

1.7.1 Auctions and Contests with loss averse players

We now study how loss aversion affects the bidding behaviour - or respectively the effort exerted - in auctions or contests. In particular we compare the effects of loss aversion in first price auctions and all pay auctions.

Consider a contest with I participants, where one prize is to be handed out. Player i's valuation of the prize is given by his signal s_i. Each contestant's effort is denoted by a_i and is interpreted as the proportion of his valuation he spends. The outcome function is given as follows:

$$\phi_i(\mathbf{a}, \mathbf{s}, \theta) = \begin{cases} 1 & \text{if } a_i s_i \geq a_j s_j \text{ for all } j \neq i \\ 0 & \text{otherwise} \end{cases} \tag{1.3}$$

We consider information structures $\{g, F(\mathbf{s}|\theta)\}$ which are scalable according to definition 3.5.1. In the scalable game of asymmetric information the probability player i assigns to winning the contest and hence the contest success function is denoted $\psi(a_i, a_{-i})$. Assuming that contestants choose constant efforts, this function is independent of s_i and is given as follows:

$$\psi_i(a_i, a_{-i}) = \int_{-\infty^n}^{\mathbf{s}} (\phi_i(\mathbf{a}, \tilde{\mathbf{s}}, \theta)|\theta) d\tilde{\mathbf{s}} \tag{1.4}$$

Note that this function is strictly increasing in a_i given a_{-i}. Given a vector of actions $\mathbf{a}$, a contestant is equally likely to win, independent of his signal.

The analysis extends to any contest success function $\psi_i(a_i, a_{-i})$ which is strictly increasing in a_i given a_{-i} and does not depend on s_i, but is not necessarily derived from a deterministic allocation rule.

In addition we assume that players are loss averse. In particular, players feel a loss, whenever their true payoff is lower than their expected payoff. This loss is given by β times the expected payoff minus the actual payoff, whenever this is positive:

$$u_i = \begin{cases} \pi_i - \beta(E(\pi_i) - \pi_i) & \text{if } \pi_i \leq E(\pi_i) \\ \pi_i & \text{otherwise} \end{cases} \tag{1.5}$$

A related definition of expectation based loss aversion is considered in Koszegi and Rabin (2006).

1.7.2 Loss Aversion in standard contests

First we consider a standard contest, where each contestant pays his effort. Player i's expected utility is denoted $V(a_i, a_{-i}|\beta)$ and is given as follows:

$$V(a_i, a_{-i}|\beta) = \psi_i(a_i, a_{-i})s_i - as_i - \beta\Big[1 - \psi_i(a_i, a_{-i})\Big]\Big[\psi_i(a_i, a_{-i})s_i\Big] \tag{1.6}$$

Note that the corresponding distribution over outcomes is scalable (see definition 4). Since we have assumed a scalable information structure (see definition 3.5.1), this is a scalable game of asymmetric information. We now show that this problem is indeed tractable.

First differentiating with respect to a_i observing that in equilibrium $\frac{\delta V(a_i, a_{-i}|\beta)}{\delta a_i} = 0$, we consider the effect of changes in the degree of loss aversion β:

$$\frac{\delta V}{\delta\beta\delta a_i} = 2\Big[\psi_i(a_i, a_{-i}) - \frac{1}{2}\Big] \tag{1.7}$$

Since V is a single-peaked function. Hence at equilibrium $\frac{\delta V}{\delta a_i} = 0$. If $\psi_i(a_i, a_{-i}) > \frac{1}{2}$, then an increase in β will increase $\frac{\delta V}{\delta a_i} = 0$. In order to remain in equilibrium a_i will increase. On the other hand if $\psi_i(a_i, a_{-i}) < \frac{1}{2}$, the opposite effect prevails and a_i will decrease. This leads to the following proposition:

Proposition 1.7.1. *In a contest where all players pay their effort and which is a scalable game of asymmetric information $\mathcal{A}(\Gamma)$, if players become more loss averse* *(ie β increases), then*

- *Players with over a half chance of winning will bid higher.*
- *Bidders with under a half chance of winning will bid lower.*

1.7.3 First price auction with loss aversion

Now consider the case of a contest, where players do not have to pay their cost and hence a generalised first price auction with loss averse players. The distribution over outcomes remains scalable and the problem is still tractable. In this case, the expected utility is given as follows:

$$\begin{aligned} V(a_i, a_j|\beta) &= \psi_i(a_i, a_j)\Big[s_i - as_i\Big] - \beta\Big[1 - \psi_i(a_i, a_j)\Big]\Big[\psi_i(a_i, a_j)(s - as_i)\Big] \\ &= \Big[(1 - a_i)\psi_i(a_i, a_j)\Big]\Big[1 - \beta\psi_i(a_i, a_j)\Big] \end{aligned}$$

Differentiating with respect to a_i and noting that in equilibrium $\frac{\delta V}{\delta a_i} = 0$ gives:

$$a_i = 1 - \frac{\psi_i(a_i, a_j) - \beta\psi_i(a_i, a_j)^2}{\frac{\delta\psi_i(a_i, a_j)}{\delta a_i}} \tag{1.8}$$

Moreover in a symmetric first price auction, the player with highest bid receives the object and $a_i = a_j$. Hence:

$$a_i = 1 - \frac{\frac{1}{2} - \beta\frac{1}{4}}{\delta\psi_i(a_i, a_i)\delta a_i} \tag{1.9}$$

An increase in β leads to an increase of the right hand side. Therefore for the first order condition to continue to hold, a_i must increase. This leads to the following result:

Proposition 1.7.2. *In a generalised first price auction which is a scalable game of asymmetric information, where players are loss averse, then: If players become more loss averse (ie β increases) they choose a higher effort (bid).*

Using propositions 1.7.1 and 1.7.2 we can determine the optimal strategy for a loss averse bidder participating in a general contest or a first price auction respectively. Although due to the improper prior, the seller's expected revenue cannot be computed under the scalable information structure, it can be calculated for any given state θ. Propositions 1.7.1 and 1.7.2 can therefore be used to determine a seller's - and buyers' - preferred auction mechanism when players are loss averse.

1.8 Analysis: Equivalence

In many applications players' preferences satisfy the independence axiom. This means that players have expected utility and their preferences can be represented by von Neumann-Morgenstern utility functions. We now provide a sufficient condition for preferences to satisfy definition 3.5.1 when players are expected utility maximisers. This expected utility representation will also be used to show the equivalence between games of asymmetric information and games of symmetric information.

We denote the von Neumann-Morgenstern utility function of player i by $u_i(\mathbf{a}, \theta, \mathbf{s})$, where $\mathbf{a}$ is the vector of players' actions, θ is the state and $\mathbf{s}$ is the vector of players' signals.

Attention is limited to utility functions which satisfy the following:

Assumption 1 (Scalable payoff structure)**.**

$$g(\theta)u_i(\mathbf{a}, \mathbf{s}, \theta) = g(\theta \oplus_H k)u_i(\mathbf{a}, \mathbf{s} \oplus_H k, \theta \oplus_H k) \textit{ for all } i \in I$$

It is clear that if a utility function satisfies assumption 1, then the corresponding preference relation over lotteries are scalable (see definition 3.5.1). Therefore utility functions that satisfy assumption 1 are a special case of the more general class of preferences studied in the previous section. Assumption 1 is satisfied when $H(\theta) = \theta$, the operator $\oplus_H$ represents $+$ and:

$$u_i(\mathbf{a}, \mathbf{s}, \theta) = u_i(\mathbf{a}, \mathbf{s} + k, \theta + k)$$

Moreover, assumption 1 is also satisfied when $H(\theta) = \ln(\theta)$, the operator $\oplus_H$ represents $\times$ and:

$$u_i(\mathbf{a}, \mathbf{s}, \theta) = \frac{1}{k} u_i(\mathbf{a}, \mathbf{s}.k, \theta.k)$$

Hence assumption 1 holds for utility functions which are (i) homogeneous of degree 0 in the log transform and (ii) homogeneous of degree 1. In particular, it is satisfied by all the examples using utility functions given in this paper. This includes (i) beauty contests and quadratic utility models where utility functions are homogeneous of degree 0 in the log transform and (ii) first price, second price and all pay auctions with risk neutral bidders where utility functions are homogeneous of degree 1. Using this assumption, we can now state the main result of this paper:

Theorem 1.8.1. *Suppose $\mathcal{A}(\Gamma)$ is a scalable game of asymmetric information and $\mathcal{S}(\Gamma)$ is a scalable game of symmetric information with the same primitives Γ. If preferences in $\Gamma^{\mathcal{S}}$ satisfy assumption 1, then the strategy profile σ^C is a Nash equilibrium of $\mathcal{A}(\Gamma)$ if and only if it is a Nash equilibrium of $\mathcal{S}(\Gamma)$.*

The proof can be found in the appendix.

This result shows that there is a correspondence between the equilibria of (i) the game $\mathcal{A}(\Gamma)$ where each player i observes some private information s_i and (ii) the game $\mathcal{S}(\Gamma)$, where all players observe some public information θ and have no private information. Therefore this result provides a deeper understanding of certain strategic situations, where the equilibrium outcomes are the same when (i) each player i observes private information s_i and (ii) players all observe the same piece of public information θ.

1.9 Applications: Symmetric information and asymmetric information

To illustrate the relevance of the link between $\mathcal{A}(\Gamma)$ and $\mathcal{S}(\Gamma)$, we now present two applications. The first application focuses on second price auctions, while the second application studies creditors bargaining in a bankruptcy situation. We then provide two examples to show that the equivalence of asymmetric and symmetric games is indeed important.

1.9.1 Second Price Auctions

First we consider a second price auction where players have valuations s_i and there is an unknown reserve price θ. In the first situation, there are two collectors interested in buying a first edition book. They are labeled $\{1, 2\}$. It could be the case that each collector knows how much he values the book (ie the value of s_i) but does not know the reserve price set by the seller (ie the value of θ). Collectors may then choose to enter the auction with $a_i = E$ or choose not to enter the auction with $a_i = NE$. To order the decisions, we assign $NE = 0$ and $E = 1$. Each collector who enters submits a secret bid (a collector who does not enter submits a bid of 0), and if the reserve price is not met then there is no sale. Crucially there is a cost to attending the auction given by c. Therefore a collector may be put off attending the auction because of the cost involved in participating. Assuming that when a collector chooses to enter he bids his valuation, the following utility function represents this situation.

$$u_i(\mathbf{a}, \theta, \mathbf{s}) = \begin{cases} s_i - \max\{s_j, \theta\} - c & \text{if} \quad a_i = P \qquad a_j = P \qquad s_i > \max\{s_j, \theta\} \\ -c & \text{if} \quad a_i = E \quad a_j = \{E, NE\} \quad s_i < \max\{s_j, \theta\} \\ 0 & \text{if} \quad a_i = NE \end{cases} \tag{1.10}$$

Secondly we consider a second price auction for oil tracts. Now consider a situation with two oil firms labeled $\{1, 2\}$. It could well be the case that the buyer knows the reserve price (denoted by θ), but does not can only estimate how much oil the tract contains and hence the value of the oil tract (denoted by s_i). Firms may then choose to enter the auction with $a_i = E$ or choose not to enter the auction with $a_i = NE$. Each firm who enters submits a secret bid (a firm who does not enter submits a bid of 0), and if the reserve price is not met then there is no sale. Again there is a cost to attending the auction of c. Therefore as before a firm may be put off attending the auction because of the cost involved in participating. This situation is represented by exactly the same utility function as above, but instead of observing s_i firms observe θ.

Note that the utility function is the same in both cases and players are risk neutral expected utility maximisers. Assuming that in both situations $g(\theta) = 1$ for all $\theta \in \mathbb{R}$ ensures assumption 1 holds. In addition assuming that the distribution of valuations such that definition 3.5.1 is satisfied, we can apply theorem 1.8.1. From the theorem it follows directly, that the two games described have the same set of constant equilibria.

Since the second game is a game of symmetric information, it is possible to average over the uncertainty to form a complete information game $\mathcal{C}(\Gamma)$. We define

$$\pi_i = P(s_i > \max\{s_j, \theta\})E[s_i - \max\{s_j, \theta\}|s_i > \max\{s_j, \theta\}]$$

to be the expected payoff of player i given that both players participate in the auction:

	E	NE
E	$\left(\pi_1 - c, \pi_2 - c\right)$	$\left(\frac{E[s_1-\theta\|s_1>\theta]}{P(s_1>\theta)} - c, 0\right)$
NE	$\left(0, \frac{E[s_2-\theta\|s_2>\theta]}{P(s_2>\theta)} - c\right)$	$\left(0, 0\right)$

Having tackled the problem using a general distribution, to fix ideas we now consider a specific example. Say s_1 is drawn uniformly from $[\theta, \theta + 6]$, while s_2 is drawn uniformly form $[\theta, \theta + 4]$. The table above now reduces to:

	E	NE
E	$(2 - c, \frac{2}{3} - c)$	$(3 - c, 0)$
NE	$(0, 2 - c)$	$(0, 0)$

If $c \leq \frac{2}{3}$, then it is a dominant strategy for each player to enter the auction. This is because player 1 receives (at worst) an expected payoff of $2 - c > 0$, while player 2 receives (at worst) an expected payoff of $\frac{2}{3} - c \geq 0$.[7] Hence an auctioneer can

[7]Despite the weak inequality it is still a dominant strategy because if player 1 chooses NE

guarantee himself revenue $R = \min\{s_1, s_2\} + \frac{4}{3}$ by setting the entry cost c to be $\frac{2}{3}$. This is better than simply running a second price auction where the auctioneer raises revenue $\min\{s_1, s_2\}$.

If $c \in (\frac{2}{3}, 3)$ only player 1 will participate in the auction. Hence the auctioneer will sell the object at the reserve price of $\theta \leq \min\{s_1, s_2\}$. Hence in this case the optimal entry cost is $\frac{2}{3}$. This analysis shows that an auctioneer can set an entry cost players are always willing to pay. Importantly the entry cost *does not jeopardise the chance that the object is sold*. This is true even though the reserve price is included in the support of all the players. This effect is driven by the fact that no player knows he has a valuation close to the reserve price and so each player is willing to pay an entry fee in the hope that he has a valuation significantly above the reserve price. This strikingly differs from the standard model, where players with low valuations are unwilling to pay entry fees.[8]

This simple example gives an indication of how the modelling tool proposed in our paper can be applied to second price auctions to help set either the entry cost or the reserve price. The next section looks at how this modelling tool can be used to uncover links between games which have been studied in the literature.

1.9.2 Bankruptcy and Bargaining

Consider a company going bankrupt. There are two senior creditors numbered $\{1, 2\}$. Creditor i is owed s_i. However there is only θ to distribute and it may be the case that $s_1 + s_2 > \theta$ and the company does not have enough money to fully repay its senior creditors.

Each creditor demands part of his money. Hence the strategy set for each player is given by $A_i = [0, 1]$, where $a_i = 1$ captures a creditor demanding all his money and $a_i < 1$ captures a creditor demanding only some of his money.

If there the company has enough money to satisfy both demands then each creditor is paid the amount he demanded (any surplus is divided between junior creditors).

then $2 - c > 0$

[8] A resulting effect of the standard model is that entry fees typically mean that the object may not be sold.

However if the company does not have enough money to satisfy both demands then creditors enter arbitration. Each creditor is awarded a fraction β_i of the surplus. However since $\beta_1 + \beta_2 < 1$, there is always an agreement that Pareto dominates disagreement.

$$u_i(\mathbf{a}, \theta, \mathbf{s}) = \begin{cases} a_i s_i & \text{if} \quad a_1 s_1 + a_2 s_2 \leq \theta \\ \beta_i s_i & \text{otherwise} \end{cases} \tag{1.11}$$

We assume a scalable information structure with $G(\theta) = \ln(\theta)$ and $g(\theta) = \frac{1}{\theta}$. Moreover, it can be checked that the preference relation is scalable with respect to this G satisfying assumption 1 and hence theorem 1.8.1 applies.

One typical situation in a bankruptcy case is where the assets of the company are unknown and players have private information about how much they are owed. This would be captured by the game $\mathcal{A}(\Gamma)$. Another situation is where the assets of the company are known, but players do not know exactly how much they will gain in arbitration. In case players choose which proportion of their claim to demand, this would be captured by the game $\mathcal{S}(\Gamma)$. This correspondence unifies much of the literature on bargaining.

1.10 Conclusion

In this paper we proposed a framework for modelling situations of asymmetric information. This framework can be used to model such situations in a tractable way and establishes a close connection between certain games of asymmetric information and games of symmetric information. The relevance of both points is illustrated using examples.

In future work we would like to use this framework to study particular situations in more detail and provide additional key applications. Such applications consider the risk estimates banks provide to the regulator as well as a general model for the formation of asset price bubbles. Furthermore an extension to multi dimensional signal spaces and action spaces may increase the relevance of this framework. In addition we are interested in comparing the scalable information structure proposed here, to other information structures typically used in the literature. In particular we would like to draw the link with the independent types assumption and consider intermediate cases. The goal is to develop a full comprehensive framework covering a range of information structures, for the general class of payoffs considered in this paper.

1.11 Appendix A: Proofs Analysis: Simplicity

1.11.1 Proof of proposition 1.6.1

Proof. Suppose the strategy profile σ^* is a Nash equilibrium of $\mathcal{C}(\Gamma)$. If $a_i^* \in \text{supp}(\sigma_i^*)$, then $(a_i^*, \sigma_{-i}^*) \succeq_i^{\mathcal{C}} (\hat{a}_i, \sigma_{-i}^*)$ for all $\hat{a}_i$.

Hence $L_i[a_i^*, \sigma_{-i}^*, 0_H] \succeq_i^{\mathcal{C}} {}_i L_i[\hat{a}_i, \sigma_{-i}^*, 0_H]$.

By the scalability of payoffs (definition 3.5.1) and the scalability of the information structure (definition 4):

$$L_i[a_i^*, \sigma_{-i}^*, 0_H] \succeq_i^{\mathcal{C}} L_i[\hat{a}_i, \sigma_{-i}^*, 0_H] \Rightarrow L_i[a_i^*, \sigma_{-i}^*, s_i^*] \succeq^{\mathcal{A}} L_i[\hat{a}_i, \sigma_{-i}^*, s_i^*]$$

Hence player i has no incentive to deviate from the strategy profile σ_i^* when he observes s_i^*.

□

1.12 Appendix B: Proofs Applications: Simplicity

1.12.1 Proof of proposition 1.7.1: Loss Aversion in an all pay contest

Proof.

$$V(a_i, a_{-i}|\beta) = \psi_i(a_i, a_{-i})s_i - as_i - \beta\Big[1 - \psi_i(a_i, a_{-i})\Big]\Big[\psi_i(a_i, a_{-i})s_i\Big]$$

Differentiating with respect to a_i gives:

$$\begin{aligned}
\frac{\delta V}{\delta a_i} &= \frac{\delta\psi_i(a_i,a_{-i})}{\delta a_i} - 1 - \beta\Big[1-\psi_i(a_i,a_{-i})\Big]\Big[\frac{\delta\psi_i(a_i,a_{-i})}{\delta a_i}\Big] + \beta\Big[\frac{\delta\psi_i(a_i,a_{-i})}{\delta a_i}\Big]\Big[\psi_i(a_i,a_{-i})\Big] \\
&= \frac{\delta\psi_i(a_i,a_{-i})}{\delta a_i} - 1 - \beta\frac{\delta\psi_i(a_i,a_{-i})}{\delta a_i}\Big[1-\psi_i(a_i,a_{-i})\Big] + \beta\Big[\psi_i(a_i,a_{-i})\Big] \\
&= \frac{\delta\psi_i(a_i,a_{-i})}{\delta a_i} - 1 + \beta\frac{\delta\psi_i(a_i,a_{-i})}{\delta a_i}\Big[2\psi_i(a_i,a_{-i}) - 1\Big]
\end{aligned}$$

$$\frac{\delta V}{\delta a_i} = \frac{\delta\psi_i(a_i,a_{-i})}{\delta a_i} - 1 + \beta\frac{\delta\psi_i(a_i,a_{-i})}{\delta a_i}\Big[2\psi_i(a_i,a_{-i}) - 1\Big]$$

Differentiating with respect to β yields:

$$\frac{\delta V}{\delta\beta\delta a_i} = 2\frac{\delta\psi_i(a_i,a_{-i})}{\delta a_i}\Big[\psi_i(a_i,a_{-i}) - \frac{1}{2}\Big]$$

Now it is assumed that V is a single-peaked function and we use the condition that $\psi_i(a_i,a_{-i})$ is strictly increasing in a_i. Hence at equilibrium $\frac{\delta V}{\delta a_i} = 0$. □

1.12.2 Proof of proposition 1.7.2: Loss aversion in a first price contest

Proof.

$$\begin{aligned}
V(a_i,a_j|\beta) &= \psi_i(a_i,a_j)\Big[s_i - as_i\Big] - \beta\Big[1-\psi_i(a_i,a_j)\Big]\Big[\psi_i(a_i,a_j)(s-as_i)\Big] \\
&= \Big[(1-a_i)\psi_i(a_i,a_j)\Big]\Big[1-\beta\psi_i(a_i,a_j)\Big]
\end{aligned}$$

$$\frac{\delta V}{\delta a_i} = \Big[(1-a_i)\psi_i(a_i,a_j)\Big]\Big[-\beta\frac{\delta\psi_i(a_i,a_j)}{\delta a_i}\Big] + \Big[(1-a_i)\frac{\delta\psi_i(a_i,a_j)}{\delta a_i} - \psi_i(a_i,a_j)\Big]\Big[1-\beta\psi_i(a_i,a_j)\Big]$$

Looking at just the terms involving β:

$$\begin{aligned}\frac{\delta V'}{\delta a_i} &= -\beta\Big[\frac{\delta\psi_i(a_i,a_j)}{\delta a_i}\Big((1-a_i)\psi_i(a_i,a_j)\Big) + \psi_i(a_i,a_j)\Big((1-a_i)\frac{\delta\psi_i(a_i,a_j)}{\delta a_i} - \psi_i(a_i,a_j)\Big)\Big] \\ &= -\beta\Big[\psi_i(a_i,a_j)^2\Big]\end{aligned}$$

Looking at terms not involving β:

$$\frac{\delta V''}{\delta a_i} = \Big[(1-a_i)\frac{\delta\psi_i(a_i,a_j)}{\delta a_i} - \psi_i(a_i,a_j)\Big]$$

In equilibrium $\frac{\delta V}{\delta a_i} = \frac{\delta V'}{\delta a_i} + \frac{\delta V''}{\delta a_i} = 0$. Hence:

$$0 = \Big[(1-a_i)\frac{\delta\psi_i(a_i,a_j)}{\delta a_i} - \psi_i(a_i,a_j)\Big] + \beta\Big[\psi_i(a_i,a_j)^2\Big]$$

Collecting terms:

$$\begin{aligned}
\frac{\beta\Big[\psi_i(a_i,a_j)^2\Big]\Big] + \psi_i(a_i,a_j)}{\frac{\delta\psi_i(a_i,a_j)}{\delta a_i}} &= \Big[(1-a_i)\Big] \\
a_i &= 1 - \frac{\beta\Big[\psi_i(a_i,a_j)^2\Big]\Big] + \psi_i(a_i,a_j)}{\frac{\delta\psi_i(a_i,a_j)}{\delta a_i}} \\
a_i &= 1 - \psi_i(a_i,a_j)\left[\frac{\delta\psi_i(a_i,a_j)}{\delta a_i}\right]^{-1}\Big[\beta\psi_i(a_i,a_j)+1\Big]
\end{aligned}$$

An increase in β leads to an increase in the right hand side of the equation (if a_i is held constant). Hence for the FOC to continue to hold a_i must increase.

□

1.13 Appendix C: Proofs Analysis: Equivalence

1.13.1 Proof of Theorem 1.8.1

Proof. We first prove this for the case where $f(s_i|\theta)$ is a discrete distribution.

Suppose $\sigma(\mathbf{a}|\mathbf{s}) = \sigma^C(\mathbf{a})$ is a constant strategy profile. Suppose also that σ is a pure strategy so that for some $\mathbf{a}^* = (a_1^*, ..., a_n^*)$, it holds that $\sigma^C(\mathbf{a}) = 1$ whenever $a_i \geq a_i^*$ for all $i \in I$ and $\sigma^C(\mathbf{a}) = 0$ otherwise. Suppose further that σ is a BNE of $A(\Gamma)$. This means that when player i has signal 0_H he has no incentive to deviate. Hence for all deviations $\hat{a}_i \in A_i$ it holds that $V_1(a_i^*, a_{-i}^*; 0_H) \geq V_1(\hat{a}, a_{-i}^*; 0_H)$ where:

$$V_1^A(a_i, a_{-i}) = \sum_{(\theta, s_{-i})} g(\theta|0_H) f_{-i}(\mathbf{s}_{-i}|\theta) u_i(a_i, a_{-i}; 0_H, \mathbf{s}_{-i}; \theta)$$

Note that:

$$g(\theta|0_H) = \frac{g(\theta)f(0_H|\theta)}{\sum_{\tilde{\theta}} g(\tilde{\theta})f(0_H|\tilde{\theta})}$$

Now define:

$$V_2^A(a_i, a_{-i}) = \sum_{(\theta, s_{-i})} g(\theta) f(0_H|\theta) f_{-i}(\mathbf{s}_{-i}|\theta) u_i(a_i, a_{-i}; 0_H, \mathbf{s}_{-i}; \theta)$$

Substituting $g(\theta|0_H) = \Big[g(\theta)f(0_H|\theta)\Big]\Big[\int g(\tilde{\theta})f(0_H|\tilde{\theta})d\tilde{\theta}\Big]^{-1}$ and multiplying each side by the constant in the second set of square brackets it follows that $V_2(a_i^*, a_{-i}^*; 0_H) \geq V_2(\hat{a}, a_{-i}^*; 0_H)$. Now define:

$$V_3^A(a_i, a_{-i}) = \sum_{(\theta, s_{-i})} f(0_H|\theta) f_{-i}(\mathbf{s}_{-i}|\theta) u_i(a_i, a_{-i}; 0_H \ominus \theta, \mathbf{s}_{-i} \ominus \theta; 0_H)$$

Note that by the extra condition imposed it follows that:

$$g(\theta)u_i(a_i, a_{-i}; \theta; 0_H, \mathbf{s}_{-i}) = g(0_H)u_i(a_i, a_{-i}; 0_H \ominus \theta, \mathbf{s}_{-i} \ominus \theta; 0_H)$$

It follows from this equation that $V_2^A(a_i, a_{-i}) = g(0_H)V_3^S(a_i, a_{-i})$ and hence $V_3^S(a_i^*, a_{-i}^*) \geq V_3^S(\hat{a}_i, a_{-i}^*)$.

Define also:

$$V_4^S(a_i, a_{-i}) = \sum_{(\hat{s}_i, \hat{s}_{-i})} f(\hat{s}_i|0_H) f_{-i}(\hat{\mathbf{s}}_{-i}|0_H) u_i(a_i, a_{-i}; \hat{s}_i, \hat{\mathbf{s}}_{-i}; 0_H)$$

Define $\hat{s}_i = 0_H \ominus_H \theta$ and $\hat{s}_j = s_j \ominus_H \theta$. Note that from the assumption that the distribution is scalable it follows that (i) $f(\hat{s}_i|0_H) = f(0_H|\theta)$ and (ii) $f(\hat{s}_i|0_H) = f(s_j|\theta)$. Using these facts and substitutions it follows that $V_3^S(a_i, a_{-i}) = V_4^S(a_i, a_{-i})$. Hence $V_4^S(a_i^*, a_{-i}^*) \geq V_4^S(\hat{a}_i, a_{-i}^*)$. This shows that $\sigma^C(\mathbf{a})$ is also a Nash equilibrium of the game of symmetric information. The reverse direction can easily be seen by repeating the steps above. Finally it is clear that the case where $f(s_i|\theta)$ is a continuous distribution (although needing more notation) can be proved along similar lines.

$\square$

1.14 Appendix D: Additional Material

1.14.1 Alternative specification of preferences: scalable actions

We show that our framework can also be used to model situations of asymmetric information, where preferences over lotteries are unchanged when scaling the actions of all players, the signals of all players and the state. For instance, consider the case of an auction, where payoffs are homogeneous of degree one: multiplying the valuations, the bids of all players by a constant their payoffs are multiplied by the same constant. If the game satisfies this alternative definition of scalable preferences and the information structure is scalable, these games are strategically equivalent to a scalable game of asymmetric information $\mathcal{A}(\Gamma)$, where $a_i = \hat{a}_i \ominus_H s_i$ and $\hat{a}_i$ is the action chosen in this alternative game. These strategies are of the form $\sigma_i(s_i) = e_i^* \oplus_H s_i$, where e_i^* is a player specific constant. These games can be studied in their original form. However the translation to the scalable game form is useful for studying the link between asymmetric information and symmetric information.

Consider a game of asymmetric information, $\mathcal{A}(\Gamma) = I, (\hat{A}_i)_{i\in I}, (\succeq_i)_{i\in I}, g, (F_i)_{i\in I}$, where $\hat{A}_i = \Theta$ for all $i \in I$ and the information structure is scalable with respect to G (see definition 3.5.1). If the game $\mathcal{A}(\Gamma) = I, (A_i)_{i\in I}, (\succeq_i)_{i\in I}, g, (F_i)_{i\in I}$

Given $\omega = (\hat{\mathbf{a}}, \theta, \mathbf{s})$, let $\omega \hat{\oplus}_G k \equiv (\mathbf{a}\oplus_H, \theta \oplus_H k, \mathbf{s} \oplus_H k)$ and let $[L\hat{\oplus}_G k](w) \equiv L(w\hat{\oplus}_G k)$. Suppose the preference relations satisfy the following definition:

Definition 8 (Alternative scalable preference relations). *A preference relation $\succeq_i$ is alternatively scalable with respect to G if whenever:*

$$L \succeq_i L'$$

then,

$$[L\hat{\oplus}_H k] \succeq_i [L'\hat{\oplus}_H k]$$

This definition says that if a player prefers lottery L to lottery L' then, when all the actions, the state and the signals are scaled up by a constant, he continues to prefer the scaled up lottery arising from L to the one arising from L'. This definition differs from the standard definition of a scalable preference structure in that all the elements of ω are scaled including the actions.

In some applications, the preference structure satisfies this alternative definition of scalable preference relations. In these cases, there exists a transformation $a_i = \hat{a}_i \ominus_H$, which redefines the action space from $\hat{A}_i = \Theta$ to A_i, such that when the actions are changed from $\hat{a}_i$ to a_i, the transformed preference relations satisfy definition 4. Formally, this can be stated as follows:

Proposition 1.14.1. *In a game $\hat{\Gamma}^{\mathcal{A}} = \{I, (\hat{A}_i)_{i \in I}, (\succeq_i)_{i \in I}, g, F\}$, if $A_i = \Theta$ for all $i \in I$ and players' preferences satisfy definition 8, there exists a strategically equivalent game $\mathcal{A}(\Gamma) = \{I, (A_i)_{i \in I}, (\succeq_i)_{i \in I}, g, F\}$: If $\sigma_i^{\mathcal{A}}$ is an equilibrium of $\mathcal{A}(\Gamma)$, then $\hat{\sigma}_i^{\mathcal{A}}$ where $\hat{a}_i = a_i \oplus_H s_i$, is an equilibrium of $\hat{\Gamma}^{\mathcal{A}}$.*

The proof follows immediately from substitutions.[9]

As an example consider the case of an auction. The true actions players take are their bids $\hat{a}_i$ in $[0, \infty)$. A first price auction clearly satisfies definition 8. However instead of considering the bids directly, we can consider the case where players choose the proportion of their valuation they want to bid $a_i = \hat{a}_i \ominus_H s_i$. Using these proportions to describe a player's preferences, these satisfy definition 3.5.1.

[9] A similar result exists for $\mathcal{S}(\Gamma)$, but it is slightly more complicated, because actions require scaling by s_i which is not observed in $\mathcal{S}(\Gamma)$.

Chapter 2

Robustness of Subgame Perfect Implementation

2.1 Introduction

This paper studies the robustness of implementation in subgame perfect equilibrium (SPE) in the fashion of Moore and Repullo (1988) and Aghion et al. (2012).

A social choice function (SCF) is said to be implemented fully, if there exists a mechanism such that the outcome prescribed by the SCF is the unique equilibrium of the mechanism in all states. Subgame perfect implementation is relevant when sequential mechanisms are used. Although the existing literature on implementation in SPE characterizes the set of SCFs which can be implemented under different informational assumptions, these papers do not provide a distinction between SCFs that are seen to be implemented in practice and those that are not. This distinction is an important aim of implementation, as in any situation it allows a social planner to fully understand the set of SCFs he can choose from.

In this paper we show that placing a very reasonable restriction on the information players have about their own preferences and on the information they have about the preferences of others, allows to distinguish between SCFs which we are seen to be implemented in practice and those that do not appear. More precisely we focus on environments where information is almost complete and introduce information perturbations where each player has more precise information about his own preferences than do other players. These perturbations are referred to as *restricted information perturbations.*

Moore and Repullo (1988) show that under complete information almost any SCF can be implemented in SPE. Taking a step away from implementation under complete information, Aghion et al. (2012) (henceforward AFHKT) show that any implementation of a non-Maskin monotonic SCF is not robust to a general class of information perturbations we refer to as *full perturbations.* Maskin monotonicity is a very restrictive requirement and is violated by many SCFs that are implemented in practice, for example firms paying a higher wage to workers with higher outside options. The result obtained by AFHKT therefore questions the usefulness of subgame perfect implementation.

In this paper we argue that typically each player is better informed about his own preferences than is any other player. We restrict attention to a class of perturbations by requiring that players know their own preferences with certainty. This is a reasonable restriction as there are many situations where each player knows his preferences, while others may be slightly uncertain.[1] One example of such a setting is that studied by Bester and Kraehmer (2012) who consider a seller making an offer to a buyer who has private information about how much he values the good.

We show that these restrictions provide a good distinction between SCFs seen to be implemented in practice and those that are not. In particular we demonstrate that under these restricted information perturbations, a wide range of SCFs can be robustly implemented, including many that are not Maskin monotonic. The class of SCFs that can be implemented robustly under the *restricted perturbations* considered here is therefore strictly larger than those that can be implemented robustly under a wider range of full perturbations.

Informally, the reason why the implementability of certain SCFs is robust to restricted perturbations but not full perturbations is the following: Under restricted

[1] Our logic also applies to cases where a player is slightly uncertain about his own preferences, as long as he is more certain about them than is any other player.

perturbations, players know their preferences with certainty and do not gain information about their own preferences from the actions of another player. Meanwhile when using full perturbations players have some uncertainty about their own preferences, and hence may update their beliefs about their own preferences after the moves of other players. In particular, in a two-stage game the result of AFHKT relies on off-equilibrium beliefs which ensure that the second-mover gains a significant amount of information about his own preferences after observing an off-equilibrium move from the first-mover. The lack of belief updating considered here leads to a much larger class of robust mechanisms under restricted perturbations.

Consider the example of a single firm and two types of workers, who differ in their outside option. A 'bad' sequential equilibrium is one where a high type worker accepts a wage that is below his outside option. These equilibria may arise under full information perturbations and rely on the fact that the worker is less informed about his own preferences than the firm. This may occasionally be the case for example when the firm has more information about the job description than the worker. However in most situations this is unlikely to hold, for instance when the worker is more informed of his preferences or outside options. Hence in many applications *restricted perturbations* are the more appropriate tool for assessing whether a certain mechanism is robust. Using this analysis, subgame perfect implementation is very robust in settings where players are confident about which allocations they value.

For most of the paper, we restrict attention to non-stochastic mechanisms where players move sequentially. This restriction is motivated by the fact that in many situations mechanisms where players move simultaneously are not feasible. For instance when bargaining a player must observe the offer made by his opponent before deciding whether to accept or reject the offer made: indeed in most bargaining models - for instance Rubinstein (1982) - players move sequentially. In contrast Baliga (1999) and Bergin and Sen (1998) study implementation in a similar setting with incomplete information and extensive form games, but where players choose their actions simultaneously. These papers show that allowing players to move simultaneously leads to much more permissive results than those presented here.

Meanwhile Corchón and Ortuno-Ortín (1995) and in a generalisation Yamato (1994) consider similar information structures where each player perfectly knows the preferences of other players in his own group but has imperfect information about players outside his group. Using Bayesian and dominant strategy implementation as equilibrium concepts they find that Nash implementation in complete information is a necessary and sufficient condition for robust implementation. In this paper we focus on a two player setting and study subgame perfect implementation which is particularly relevant in sequential move games.

Our main result relates to the concept of *exact implementation* studied by Moore and Repullo (1988) as well as Abreu and Matsushima (1994). The term exact implementation in a setting with information perturbations is used to mean that the desired allocation is always implemented whenever players observe correct signals about the state. The main result of our paper then proves a sufficient condition for a SCF to be exactly implementable with restricted information perturbations. In particular we show that any SCF which can be implemented in a two-stage sequential move game in complete information can be implemented exactly with restricted information perturbations. Moreover requiring two stage implementation is more permissive than requiring Maskin monotonicity, but more restrictive than requiring only three stage implementation.

Since the necessary and sufficient conditions for two stage implementation do not provide great insight, the relevance of two stage implementation is illustrated using a number of examples. Many standard settings of principal agent interaction proceed in two stages, where the principal offers a contract. The agent can reject the contract, accept it - or in some cases - choose an action. Indeed, the examples given in this paper can be interpreted as classic principal agent settings. More precisely, the analysis can be be interpreted as studying the robustness of the outcome of principal agent interactions to small levels of asymmetric information.

Finally, we consider the weaker concept of virtual implementation studied by Abreu and Sen (1991). Virtual implementation with information perturbations requires that the desired allocation is implemented almost always, but does not exclude the possibility for the wrong allocation to be occasionally implemented even when players observe the correct signals. In a deviation from most literature we do not consider virtual implementation using a stochastic element in the mechanism.[2] Instead we follow an approach introduced by Serrano and Vohra (2010) and allow players to choose mixed strategies. We say that an SCF is virtually implementable when in the only equilibrium of the game with information perturbations, players choose mixed strategies, such that the outcome prescribed by the SCF is reached almost always and the probability with which any type chooses a different path becomes arbitrarily small when the information perturbations tend to zero.

Using an example, we show that requiring only virtual implementation some SCFs are robust to restricted information perturbations, although they are not robust when exact implementation is required. This argument shows that the set of SCFs that can be considered robust to information perturbations become larger when considering weaker concepts of implementation. The decision of which concept is appropriate may depend on the situation one has in mind.

The remainder of the paper proceeds as follows. In section two we provide an example to illustrate the differences between implementability under complete information, full perturbations and restricted perturbations respectively, as well as present the intuition behind these differences. Section three introduces the model and formal definitions. The sufficient condition for robust implementation under restricted perturbations is presented in section four. In section five we consider the case of virtual implementation. Section six concludes.

2.2 Example

Suppose a firm (P) is bargaining with a worker (A). There are two states of the world $\Theta = \{L, H\}$, which represent the fact that workers may either be high type (H) or low type (L). The probability that the worker is high type is $\alpha_H \in (0, 1)$, while the probability that the worker is low type is $\alpha_L = 1 - \alpha_H$. There are three

[2] This approach is often criticised, because implementation relies on the mechanism designer committing to occasionally implement an allocation that he knows is not Pareto efficient at the point of implementing it.

outcomes $X = \{w_H, w_L, d\}$. First a high wage w_H may be agreed, secondly a low wage w_L may be agreed and thirdly a default option d may be reached. Both types of workers are equally productive when working for the firm and so the preferences of the firm do not depend on the type of the worker. The firm prefers to pay a low wage rather than a high wage, and prefers to pay a high wage rather than failing to make an agreement:

$$\text{Firm's preferences:} \quad u_P(w_L;\theta) > u_P(w_H;\theta) > u_P(d;\theta) \quad \text{for } \theta \in \{L, H\}$$

Meanwhile, all workers prefer the high wage to any other alternative. However, low type workers prefer to receive the low wage rather than the outside option, while the high type workers prefer the outside option to the low wage. Therefore the preferences of each type of worker are given as follows:

$$\begin{array}{lll} \text{Low type's preferences:} & u_A(w_H;\theta) > u_A(w_L;\theta) > u_A(d;\theta) & \text{for } \theta = L \\ \text{High type's preferences:} & u_A(w_H;\theta) > u_A(d;\theta) > u_A(w_L;\theta) & \text{for } \theta = H \end{array}$$

All of the above is commonly known. Players negotiate according to the following two-stage sequential move bargaining procedure. In the first stage the firm makes an offer $w \in \{w_L, w_H\}$, and then in the section stage the worker chooses to accept (Y) or decline (N) the offer. If the worker accepts the wage offer this agreement is made, and otherwise the default option is reached. The extensive-form version of this game is given in Figure 1.[3]

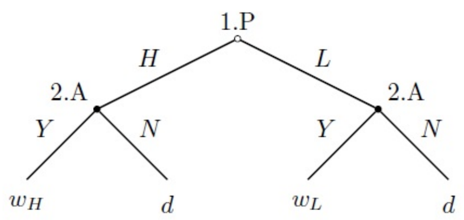


Figure 2.1. Two stage mechanism.

We analyse this game under three different information structures. In the first case

[3]Each node is an information sets and there are no moves by nature, as we assume that workers are born with their preferences.

we consider complete information, where both players know the worker's type. In the second and third case, we assume that one player knows the worker's type, while the other receives a signal $s \in \{s_L, s_H\}$ which is highly correlated with the worker's type. More precisely after observing a signal s_L the probability of the worker being low type is equal to $(1-\epsilon)$, while after observing a signal s_H the probability of the worker being high type is equal to $(1-\epsilon)$. After receiving such a signal a player is highly confident - although not completely sure - about the worker's type: in this case we say the worker's type is ϵ-known. Throughout the example it is assumed that $\epsilon > 0$ and ϵ is sufficiently small. A more formal approach is taken in the next section.

Complete information

First consider the case of complete information, where the worker's type is commonly known. In this case there is a unique SPE, where on the equilibrium path the firm offers the low type worker the low wage, the firm offers the high type worker the high wage and all offers are accepted. Off the equilibrium path, low type workers accept a high wage and high type workers reject a low wage. Therefore in complete information this mechanism implements a SCF $f(\theta)$, where $f(L) = w_L$ and $f(H) = w_H$. Note that this SCF is not Maskin monotonic, since both types of workers prefer a high wage to a low wage and yet only the high type workers receive a high wage while the low type workers receive a low wage. Formally Maskin monotonicity is defined as follows:

Definition 9 (Maskin monotonicity)**.** *An SCF ψ is Maskin monotonic, if for all $\theta, \theta' \in \Theta$:*

$\psi(\theta) = x$ *and* $\theta' \in L_i(x, \theta_i)$ *for* $i = A, B$ *imply* $\psi(\theta') = x$

where $L_i(x, \theta_i)$ is the lower contour set of player i with preferences θ_i at allocation x.

An information perturbation where workers know their own preferences

Secondly consider the case where the worker's type is known by the worker and ϵ-known by the firm. Since the worker knows his own type, high type workers always reject the low wage, while low type workers always accept it. Given ϵ is sufficiently small it follows that:

$$\alpha_L(1-\epsilon)\Big(u_P(w_L;\theta)-u_P(w_H;\theta)\Big) > \alpha_H\epsilon\Big(u_P(w_H;\theta)-u_P(d;\theta)\Big)$$

The left hand side represents the firm's gains when offering a low wage rather than a high wage to a low type player having received a signal s_L which was correct. Meanwhile the right hand side denotes the losses that the firm incurs when offering a low wage - which is rejected - rather than a high wage after an incorrect signal s_L. If ϵ is sufficiently small and the signal is sufficiently reliable, it is clear that the gains from offering a low wage outweigh the loss of occasionally reaching the default after an incorrect signal. It follows that there is a unique sequential equilibrium where the firm offers a low wage after observing a signal s_L and a high wage after observing a signal s_H. Note that the unique sequential equilibrium is very close to the complete information SPE. Hence this mechanism can be considered robust to those information perturbations where the worker knows his own preferences.

An information perturbation where workers do not know their own preferences

Finally, consider the case where the worker's type is known by the firm and ϵ-known by the worker. In this case there are two distinct sequential equilibria. First there is a separating equilibrium, which is almost outcome-equivalent to the complete information SPE. In the first stage the firm nearly always offers a high type worker the high wage and a low type worker the low wage. Then in the second stage the workers always accept if they receive a high wage or if they receive a low wage and have a low signal. They play mixed strategies when the firm offers a low wage and they receive a high signal. If ϵ is small this third case happens rarely, and the complete information outcome is nearly always reached. In this 'trusting' sequential equilibrium, workers believe the firm is very likely to have made the appropriate offer unless they have reason to believe otherwise.

However, there is also another pooling equilibrium which leads to a very different outcome. In the first stage the firm offers all workers the high wage, and in the second stage all workers accept. To ensure that this is indeed a sequential equilibrium, it is assumed that workers have the following off-equilibrium beliefs: if the firm makes a low offer (which does not happen in equilibrium), then the worker believes he is very likely to be a high type regardless of his initial signal. This means that the off-equilibrium beliefs are such that the firm's off-equilibrium move is *much more informative than the worker's original signal.* Therefore when a worker who has received a low signal s_L receives a low offer w_L, he believes there is a significant chance that he is high type and rejects the offer. In this 'suspicious' pooling equilibrium workers do not believe that the firm has made the appropriate offer when the firm makes an off-equilibrium move. These suspicious off-equilibrium beliefs sustain what AFHKT refer to as a 'bad' sequential equilibrium.

AFHKT prove that any mechanism implementing a non-Maskin monotonic SCF in complete information is not robust to certain information perturbations. This example suggests that this result relies on the fact that players learn about their own preferences from the actions of other players. The main result of this paper formalises this. We show that bad sequential equilibria arise precisely in the case where the second mover significantly updates his belief about his own preferences from observing the other player's move. We prove that any SPE implementation in complete information which uses a two stage sequential mechanism is robust to those information perturbations where players remain certain of their own preferences. This shows that many SPE implementations in complete information are robust to the class of perturbations which are most relevant for many situations.

2.3 The model

There are two players $i = \{A, B\}$ and the payoff type of each player is denoted by $\theta_i \in \Theta_i$. The state is given by the pair of payoff types $\theta = (\theta_A, \theta_B) \in \Theta_A \times \Theta_B = \Theta$. We let X denote the set of allocations, while players' Bernoulli utilities are denoted by $u_i(x; \theta_i)$. These utilities depend only on the eventual allocation $x \in X$ and the player's type θ_i. It is assumed that the state space Θ and the set of outcomes X are finite. A complete information SCF f is a one to one mapping from a state to an outcome, $f : \ \Theta \mapsto X$.

Before any move is made, player A observes a signal $s^A = (s^A_A, s^A_B) \in S^A$ and player B observes a signal $s^B = (s^B_A, s^B_B) \in S^B$ where s^i_j is a signal about player $j's$ preferences. We identify the signal sets with the state space so that $S^A = S^B = \Theta$. Signals are drawn from a common prior described by $\nu \in V$, where $\nu : \Theta \times S^A \times S^B \mapsto [0, 1]$ and $\sum \nu = 1$.

We restrict our focus to extensive form mechanisms Γ with a finite number of stages where players move sequentially and every move is immediately and perfectly observed by the other player. Without loss of generality it is assumed that player A moves first, players move alternately and the number of stages is $2N$ for some $N \in \mathbb{N}$).

In any stage n, if n is odd then player A chooses a strategy $\sigma_{A,n} \in \Sigma_{A,n}$, while if n is even then player B chooses a strategy $\sigma_{B,n} \in \Sigma_{B,n}$. Therefore in the first stage player A makes a move, in the second stage player B moves and so on. Let $\sigma_A = (\sigma_{A,1}, \sigma_{A,3}....\sigma_{A,2N-1})$ and $\sigma_B = (\sigma_{B,2}, \sigma_{B,4}....\sigma_{B,2N})$ denote a possible set of strategies for player A and player B respectively. Furthermore let $\sigma = (\sigma_A, \sigma_B)$, and write $\Gamma(\sigma) \in X$ to mean the allocation implemented when players choose strategies σ. It is assumed that all strategy sets $\Sigma_{A,n}$ and $\Sigma_{B,n}$ are finite.

Players may condition their strategies on their signal and previously observed moves. Hence a strategy profile $h_{i,n}$ at stage n for player i maps a vector $(s^i, \sigma_{i,1}, \sigma_{i,2}, ..., \sigma_{i,n-1})$ to a strategy σ_n. A complete strategy profile h_i for player i denotes a set of strategy profiles for each stage where that player moves. Hence the strategy profile $h = (h_A, h_B)$ is a subgame perfect equilibrium (SPE) of the complete information game Γ if players have no incentive to deviate from this strategy profile.

Players initially form their beliefs based on their signal and the initial common prior. As the game progresses, players may update their beliefs after the move of an opponent. A belief profile $\phi_{i,n}$ for player i at stage n maps a vector $(s^i, \sigma_{i,1}, \sigma_{i,2}, ..., \sigma_{i,n-1})$ to a prior ν. A complete belief profile ϕ_i denotes a set of belief profiles for every stage, and $\phi = (\phi_A, \phi_B)$ denotes a pair of such belief profiles. The pair (h, ϕ) is a sequential equilibrium (SE) induced by the game (Γ, v) if ϕ represents a set of consistent beliefs given that (i) players are playing according to the strategy profile h and (ii) given their beliefs ν players have no incentive to deviate from the strategy profile h in any information set.[4]

2.3.1 Three informational environments

We now outline three possible restrictions on the prior ν which capture three different informational environments. First consider a complete information environment where players are certain of each others preferences. This is only the case when players always receive the correct signal about their own preferences and the preferences of their opponent. Hence we say that ν^0 is a complete information prior if ν puts probability 1 on $s^A = s^B = \theta$.

Definition 10 (Complete information)**.** *The prior ν^0 is a complete information prior, if and only if*

$$\sum_{\theta \in \Theta} \nu^0(\theta, \theta, \theta) = 1$$

Secondly consider the environment where both players observe a highly reliable signal about the preferences of both players as studied by AFHKT. In particular suppose that the reliability of the signal is such that a player is misinformed about either the preferences of his opponent or his own preferences with a probability lower than ϵ. Therefore $s^A = \theta$ and $s^B = \theta$ with probability greater than $1 - 2\epsilon$, and hence we define a full (ϵ)-perturbation as follows:

Definition 11 (Full (ϵ)-perturbations)**.** *The prior ν^ϵ is a full (ϵ)-perturbation if and only if*

$$\sum_{\theta \in \Theta} \nu^\epsilon(\theta, \theta, \theta) > 1 - 2\epsilon$$

Finally consider an environment where players are certain of their own preferences and observe a highly reliable signal about the preferences of the other player. Suppose that players are misinformed about the preferences of his opponent with a probability lower than ϵ. As before, since players are almost always correctly informed about both their preferences and their opponent's preferences $s^A = \theta$ and $s^B = \theta$ with probability $1 - 2\epsilon$. However since players are certain of their own

[4]This definition follows Aghion et al. (2012) who provide a formal definition of a sequential equilibrium in these multistage games in their online appendix.

preferences there is an additional requirement, since both $s_A^A = \theta_A$ and $s_B^B = \theta_B$ with probability 1. Hence a prior ν^ϵ with restricted (ϵ)-perturbations is defined as follows:

Definition 12 (Restricted-(ϵ) perturbations)**.** *The prior ν^ϵ is a restricted (ϵ)-perturbation if and only if*

1. *ν^ϵ is a full (ϵ)-perturbation*
2. *If $s_A^A \neq \theta_A$, then $\nu^\epsilon(\theta, s^A, s^B) = 0$*
3. *If $s_B^B \neq \theta_B$, then $\nu^\epsilon(\theta, s^A, s^B) = 0$*

Finally define V_C to be the set of complete information priors, V_F^ϵ to be the set of full (ϵ)-perturbations and V_R^ϵ to be the set of restricted (ϵ)-perturbations. Note that $V_C \subset V_R^\epsilon \subset V_F^\epsilon$. The next two sections investigate under what conditions exact implementation and virtual implementation are robust to restricted (ϵ)-perturbations.

2.4 Exact implementation

We now give a definition of exact implementation in an environment with information perturbations. We say that a SCF f is robustly implementable with information perturbations if - when perturbations are sufficiently small - the desired outcome is implemented with probability one whenever players receive the correct signals.[5] Under information perturbations, the definition of exact implementation can be extended as follows:

Definition 13. *A mechanism Γ exactly implements a SCF $f : X \mapsto \Theta$ with restricted (full) perturbations if and only if given any complete information prior $\nu^0 \in V_C$ and any sequence of priors $\{\nu^\epsilon\}_{\epsilon>0}$ whenever*

1. *$\nu^\epsilon \in V_R^\epsilon$ ($\nu^\epsilon \in V_F^\epsilon$)*

[5] Note that the standard definition of exact implementation requires the desired allocation to be implemented with probability one in all cases. Under information perturbations this definition leads to trivial results, since clearly the wrong allocation will arise when players receive the wrong signals. For the analysis to be sensible, the definition is adapted to allow for other outcomes in the rare case, where players receive wrong signals.

2. *The sequential equilibrium* $(\sigma^\epsilon, \phi^\epsilon)$ *is induced by the game* (Γ, ν^ϵ)

then there exists some $\bar{\epsilon}$ *such that* $\Gamma(\sigma^\epsilon) = f(\theta)$ *whenever i)* $\epsilon < \bar{\epsilon}$ *and ii)* $s^A = s^B = \theta$

Using this definition, the main result of AFHKT applies in our setting:

Theorem 2.4.1 (AFHKT)**.** *An SCF f can be robustly implemented with full perturbations if and only if*

1. *f is Maskin-monotonic*

2. *f is implementable in a complete information setting*

This result holds in a very general setting with $n \geq 2$ players, where moves may be either sequential or simultaneous. It relies on the fact that in extensive form games with several stages, additional equilibria can be formed by choosing off-equilibrium beliefs judiciously. We discussed an example of an additional bad equilibrium that arises when full perturbations are considered in the previous section. It follows that using additional stages does not increase the number of SCFs that can be implemented. As shown by AFHKT, certain small information perturbations can reduce the power of sub-game perfect implementation significantly.

However if we rule out the possibility that players are mistaken about their own preferences and only consider this smaller class of restricted perturbations, the situation is not nearly so bleak. Our example has already shown that the implementability of some SCFs are robust to restricted perturbations and not full perturbations. We now generalise this result and give a sufficient condition for exact implementation under restricted perturbations.

2.4.1 Sufficient condition

In this section we introduce a sufficient condition for exact implementation with restricted information perturbations which is significantly weaker than Maskin-monotonicity. This shows that restricting the set of information perturbations in an intuitive way significantly increases the set of SCFs that are robustly implementable. We first make the following definition:

Definition 14 (F_2)**.** *An SCF $f \in F_2$ if it can be implemented under complete information by a two stage mechanism with sequential moves.*

We now state our sufficient condition for robust implementation with restricted information perturbations:

Theorem 2.4.2 (Sufficiency)**.** *If an SCF $f \in F_2$, then it can be robustly implemented with restricted information perturbations.*

In order to prove this result we first characterize the SPEs under full information in two stage sequential move games. We have to show that the strategy profile used in any sequential equilibrium with sufficiently small restricted information perturbations coincides with a SPE in complete information. It is easy to show that the second mover - assuming he receives the correct signal - chooses his strategy in the same way as he does under complete information, because when making his decision, the second mover has not updated his preferences and simply chooses the allocation he likes most. Given that the second-mover behaves as he does under complete information, it is then possible to show that the first-mover also behaves as he does under complete information as long as his signal is correct and perturbations are sufficiently small. The complete proof can be found in the appendix.

2.4.2 Comparison with complete information

In order to illustrate that restricted perturbations provide an appropriate criterion for distinguishing between SCFs which are seen to be implemented in practice and those that are not, we now provide a comparison with the case of complete information. We show that robustness to restricted perturbations is more restrictive than implementation under complete information.

We consider the canonical mechanism introduced by Moore and Repullo (1988). Although this mechanism can be used to implement a wide-range of SCFs under complete information, it is not robust to restricted perturbations. More precisely there exist SCFs which can be exactly implemented using this mechanism under complete information, but cannot exactly be implemented under restricted perturbations. Hence exact implementation under restricted perturbations is a more

Table 2.1. Example: Simple three stage mechanism: Implementable under complete information, not implementable under restricted perturbations

Preferences	
Firm: $\theta \in \{L, H\}$	$u_P(w_L;\theta) > u_P(x_H;\theta) > u_P(w_H;\theta) > u_P(x_L;\theta)$
Low type: $\theta = L$	$u_A(w_H;\theta) > u_A(w_L;\theta) > u_A(x_L;\theta) > u_A(x_H;\theta)$
High type: $\theta = H$	$u_A(w_H;\theta) > u_A(w_L;\theta) > u_A(x_H;\theta) > u_A(x_L;\theta)$

restrictive criterion than exact implementation under complete information. In particular many SCFs that require complex mechanisms to be implemented under complete information can not be implemented when allowing for restricted information perturbations.

This is illustrated using the following example. Again consider a setting where a firm denoted by P wants to hire a a worker denoted by A. The worker may be a high type or a low type. In this example there are two outside options denoted x_H and x_L respectively. The players' preferences are given in Table 1. Now consider the mechanism represented in Figure 2.

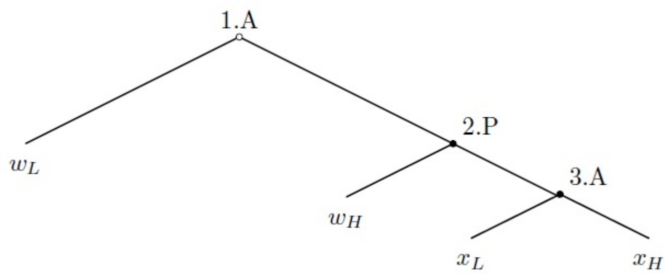


Figure 2.2. Moore and Repullo mechanism

Under complete information this Moore and Repullo mechanism implements the SCF where the high type worker receives w_L and the low type worker receives w_H. However note that the separating equilibrium implemented under complete information is not robust to restricted information perturbations. If the firm believes that it faces a high type whenever a worker starts by choosing the branch on the right, the firm will react by offering the worker the choice between the two outside options. This creates a 'bad' pooling sequential equilibrium in which all workers receive w_L. The SCF where the low type worker receives w_H and the high type worker receives w_L, is therefore not robust to restricted information perturbations. Hence this shows that the canonical Moore-Repullo mechanism is not robust to restricted perturbations.[6].

Other examples of two stage sequential move mechanisms seen in practice include a decision on a public good, where one agent announces how much he is willing to contribute, before a second agent decides to raise the amount to the critical threshold or to not contribute. Alternatively one can think of a principal agent setting, where the principal offers a menu of contracts and the agent chooses his preferred contract.

One should note that implementability in two stages under complete information is sufficient for exact implementation with restricted perturbations, but is not necessary. In the appendix we present an example of an SCF that can be exactly implemented in three stages with restricted perturbations but not in two stages.[7]

2.5 Virtual Implementation

In this section we show that the range of SCFs that are robust to restricted information perturbations becomes even larger, when considering the weaker concept of virtual implementation. Formally virtual implementation requires that for each $\epsilon > 0$ there exists a nearby game Γ^ϵ such that in any sequential equilibrium of this game the desired outcome is obtained with probability greater than $1 - \epsilon$. This is weaker than the concept of exact implementation considered previously,

[6]Note that this does not prove that the SCF cannot be implemented robustly. But it cannot be implemented robustly using the mechanism suggested by Moore and Repullo (1988)

[7]However, these examples are rare and difficult to construct. In particular the example we present is such that by allowing for simultaneous move in the first stage and then allowing one of the players to move again in the second stage, the SCF can be implemented in two stages. Hence by weakening condition $F2$ to implementability in two stages where the first stage allows for simultaneous moves, while only one player moves in the second stage.

since we now allow for the possibility that the desired outcome is occasionally not implemented even in cases when both players receive the correct signals. More precisely:

Definition 15. *A mechanism* Γ *virtually implements an SCF* $f : X \mapsto \Theta$ *with restricted (full) perturbations if and only if given any* $\delta > 0$, *any complete information prior* $\nu^0 \in V_C$ *and any sequence of priors* $\{\nu^\epsilon\}_{\epsilon>0}$ *whenever*

1. $\nu^\epsilon \in V_R^\epsilon$ $(\nu^\epsilon \in V_F^\epsilon)$

2. *The sequential equilibrium* $(\sigma^\epsilon, \phi^\epsilon)$ *is induced by the game* (Γ, ν^ϵ)

then there exists some $\bar{\epsilon}$ *such that* $P\Big(\Gamma(\sigma^\epsilon) = f(\theta)\Big) > 1 - \delta$ *whenever* $\epsilon < \bar{\epsilon}$

Most previous work on virtual implementation - see Serrano and Vohra (2010) for an exception - considers stochastic mechanisms where in equilibrium players play according to pure strategies. In these cases the slight uncertainty over the eventual outcome is caused by the stochasticity of the mechanism. In contrast, in the examples considered below slight uncertainty over the eventual outcome is caused by the fact that players do not play pure strategies, but rather play *almost pure* strategies, allowing them to deviate from the main strategy prescribed for their type occasionally.

Virtual implementation under restricted perturbations is less permissive than virtual implementation under complete information, while being more permissive than exact implementation under restricted perturbations. To show the first part of this claim it is sufficient to consider the canonical Moore-Repullo mechanism analysed above. It can immediately be seen that this mechanism - and hence the canonical mechanism - is not robust to restricted perturbations even when considering the weaker criterion of virtual implementation. This follows from the fact that this mechanism has a pooling equilibrium, as explained in the previous section. Whenever 'bad' sequential equilibria arise from pooling, both virtual implementation and exact implementation fail.

To show the second part of this claim we provide an example of an SCF which cannot be robustly implemented under restricted perturbations if exact implementation is required, but is robust when requiring only virtual implementation.

Table 2.2. Complex three stages: Virtually implementable under restricted perturbations not exactly implementable under restricted perturbations

Preferences	
Firm: $\theta \in \{L, H\}$	$u_P(y_L;\theta) > u_P(w_L;\theta) > u_P(x_H;\theta) > u_P(w_H;\theta) > u_P(x_L;\theta) > u_P(y_H;\theta)$
Low type: $\theta = L$	$u_A(w_H;\theta) > u_A(w_L;\theta) > u_A(x_L;\theta) > u_A(x_H;\theta) > u_A(y_L;\theta) > u_A(y_H;\theta)$
High type: $\theta = H$	$u_A(w_H;\theta) > u_A(w_L;\theta) > u_A(x_H;\theta) > u_A(x_L;\theta) > u_A(y_H;\theta) > u_A(y_L;\theta)$

This difference follows from the fact that exact implementation requires the complete information allocation to be implemented whenever both players receive the correct signal. Virtual implementation allows rare occasions where players deviate from their complete information strategy in which case a different allocation is implemented despite both players receiving the correct signal. Robust virtual implementation requires these 'differences' to become increasingly rare as signal precision increases. An example of such a setting is discussed below.

2.5.1 Comparison with exact implementation

We now give an example of an SCF which can be virtually implemented robustly, but cannot be exactly implemented robustly. Note also that the example is constructed such that the SCF can be virtually implemented robustly using a three stage mechanism, even though it cannot be virtually implemented using a two-stage mechanism.

Let $\Theta = \{L, H\}$, $X = \{w_L, w_H, x_H, x_L, y_H, y_L\}$ and consider the preference profile given in Table 2.

Now consider the SCF $f : \Theta \mapsto X$, where $f(H) = w_L$ and $f(L) = w_H$. This SCF is implementable using restricted perturbations but it is not implementable in a two stage sequential move mechanism in complete information.

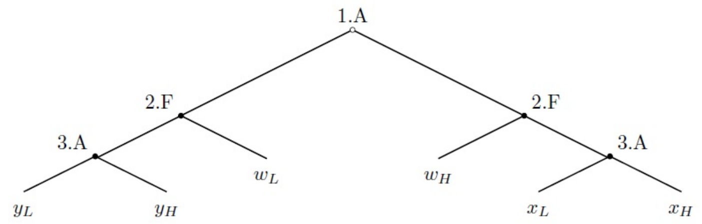


Figure 2.3. Complex three stage mechanism

To show that this SCF can be virtually implemented using restricted perturbations, consider the mechanism represented in Figure 3. This mechanism virtually implements the SCF described above both under complete information and with restricted perturbations. The extra off equilibrium outcomes y_L and y_H ensure that the bad sequential equilibrium that arises in the three stage example described in the previous section does not arise here. Note that in complete information this mechanism implements the allocation w_H if the worker is type L and w_L if the worker is type H.

When restricted information perturbations are introduced, the mechanism fails to implement this SCF exactly. To see this, consider the following equilibrium. Define m_L to be the proportion of low types and m_H to be the proportion of high types. Suppose perturbations happen with probability at most ϵ and that ϵ is sufficiently small. Finally choose mixing probabilities α and β such that the following equations are satisfied:

$$u_P(w_H) = (1-\alpha)m_H\nu(s^H|\theta_H)u_P(x_H) + m_L\nu(s^H|\theta_L)u_P(x_L)$$

$$u_H(w_L) = \Big[\nu(s^L|\theta_H) + \beta\nu(s^H|\theta_H)\Big]u_H(w_H) + (1-\beta)\nu(s^H|\theta^H)u_H(x_H)$$

In the first stage all low types choose the right branch. Meanwhile high types mix, with a proportion α choosing the left branch and a proportion $(1-\alpha)$ choosing the right branch. In the second stage if the worker chose the left branch the firm always chooses w_L. Meanwhile if the worker chose the right branch and the firm observes a signal s^L the firm always chooses w_H. Finally if the worker chose the right branch and the firm observes a signal s^H the firm mixes: with probability β the firm chooses w_H while with probability $(1-\beta)$ the firm proceeds to the third stage. In the third stage a high type worker chooses x_H or y_H while a low type worker chooses x_L or y_L.

It can be easily checked that the strategy profile above outlines a SPE whenever $\epsilon > 0$. In the appendix it is proved that this is indeed the unique SPE. Note that in the first round high types mix between choosing the left branch and the right branch, and so this mechanism does not exactly implement the desired SCF under restricted perturbations. However as $\epsilon \to 0$, then $\alpha \to 1$ where α denotes the fraction of high types who choose the left branch in the first round. This - together with the fact that the SPE outlined above is unique - shows that this mechanism does virtually implement the desired SCF under restricted perturbations. In particular if perturbations are sufficiently small, then the proportion of high types imitating low types can be made to be arbitrarily small. Hence the desired allocation is reached in almost all cases.

This example shows that when exact implementation is prevented by the behaviour of a small proportion of types, allowing players to mix with small probabilities, virtual implementation (as defined above) may still be possible. Note that as the precision of the signal increases, the proportion of players deviating from the complete information equilibrium becomes small. On the one hand - as shown in the previous section - exact and virtual implementation under restricted perturbations coincide when implementation is prevented by the creation of fully pooling 'bad' sequential equilibria. On the other hand, there exist other cases - particularly when perturbations only slightly change equilibrium outcomes - where virtual implementation is more permissive than exact implementation.

2.6 Discussion

The central message of this paper is that the power of SPE implementation depends on the relevant set of information perturbations and the strength of implementation required. At one extreme, if information perturbations are irrelevant and

Table 2.3. Summary (Example 3 can be found in the appendix)

	Exact Implementation	Virtual Implementation
Full Perturbations	Maskin Monotonic	Maskin Monotonic
Restricted Perturbations	Two-stage mechanisms and Example 3	Also Example 2
Complete Information	Condition C	Condition C

there is complete information, a wide range of SCFs can be implemented using Moore-Repullo mechanisms. Meanwhile, at the other extreme, if full perturbations are relevant, then AFHKT show that only Maskin-monotonic SCFs can be implemented. In this paper we have considered the intermediate case of restricted perturbations and provide results which lie somewhere between these two extremes. These results are summarised in Table 3.

The exact power of implementation under restricted perturbations depends on whether virtual implementation or exact implementation is required. One argument for considering virtual implementation is that the definition of exact implementation already allows for mistakes in the rare case when players receive the wrong signal. Hence the concept of exact implementation given here is already - in some sense - a restricted type of virtual implementation, and so it seems natural to instead consider the full version of virtual implementation instead. Meanwhile, an argument for considering exact implementation is that it requires players to follow pure strategies, which are more intuitive than the *almost* pure strategies players follow when considering virtual implementation.

There are two ways in which the results presented here could be easily extended. First the sufficiency result stated here can be extended to an n-player framework where each player moves exactly once. One extra restriction would be necessary: players who move earlier must not be able to communicate information about the preferences of any player who moves later. The proof would be very similar to the two-player case, albeit with extra notation.

The second extension involves considering a class of perturbations wider than those considered in this paper, but still more restricted than than full information restrictions. Note that the formation of 'bad' sequential equilibria relies on players changing their beliefs about their own type to a significant extent. Therefore the results above are also robust to a more general class of restricted perturbations. In particular consider the case where the second-mover receive a signal about their own preferences which is highly (but not perfectly) reliable, while the first-mover receives a significantly less reliable signal. In these cases the second-mover is much more informed than the first-mover, and hence only updates his beliefs about his own preferences by a small amount. This ensures 'bad' sequential equilibria cannot be formed, and that two-stage implementations continue to be robust.

2.7 Appendix

2.7.1 Proof of Proposition 2.4.2

Before proving this theorem we introduce some additional notation and definitions. We use $h_B(s^B, \sigma_A) \in \Sigma_B$ to denote the strategy chosen by player B when he observes signal s^B and player A has chosen strategy σ_A. Hence, $h_B \in H_B$ is a strategy profile of player B, where H_B is the set of all such profiles.

Meanwhile $h_A(s^A, h_B) \in \Sigma_A$ denotes the strategy chosen by player A when he observes signal s^A and expects player B to play according to strategy profile h_B. Hence $h_A \in H_A$ denotes a strategy profile determining the choice of player A when he observes a certain signal and has a certain belief about the strategy profile of player B. H_A is the set of all such strategy profiles. We now define H_B^* and $H_A^*(h_B)$, which denote the possible SPE strategy profiles that occur in a complete information setting:

Definition 16. $h_B \in H_B^*$ *if and only if for all* σ_A, *for all* θ *and for all* $\hat{\sigma}_B \in \Sigma_B$

$$u_B(\Gamma(\sigma_A, h_B(\theta, \sigma_A)); \theta_B) \geq u_B(\Gamma(\sigma_A, \hat{\sigma}_B); \theta_B)$$

Definition 17. $h_A \in H_A^*(h_B)$ *if and only if for all* θ *and for all* $\hat{\sigma}_A \in \Sigma_A$

$$u_A(\Gamma(\sigma_A, h_B(\theta, \sigma_A)), \theta_A) \geq u_A(\Gamma(\hat{\sigma}_A, h_B(\theta, \hat{\sigma}_A)), \theta_A)$$

In a complete information setting with the complete information prior ν, the following proposition is immediately implied by the definitions above:

Proposition 2.7.1. (h_A, h_B) *denote a SPE of* Γ *iff* $h_B \in H_B^*$ *and* $h_A \in H_A^*(h_B)$

This characterizes the SPEs under full information in two stage sequential move games. Note that any sequential move game with finite strategy sets has at least one equilibrium. Hence in order to prove 2.4.2 it is sufficient to show that the strategy profile used in any sequential equilibrium with sufficiently small restricted information perturbations coincides with a SPE in complete information.

To do this consider a game with restricted information perturbations (Γ, ν^ϵ) and corresponding sequential equilibrium strategy profiles $(h^\epsilon_A, h^\epsilon_B)$. It is sufficient to prove that for some $\bar{\epsilon} > 0$, $h^\epsilon_B \in H^*_B$ and $h^\epsilon_A \in H^*_A(h_B)$ whenever $\epsilon \leq \bar{\epsilon}$. The proof is now split into two parts.

First we prove that $h^\epsilon_B \in H^*_B$. This follows from the fact that player B knows his own preferences with certainty and hence in response to player A's move chooses his preferred alternative. [8]

Secondly we prove $h^\epsilon_A \in H^*_A(h^\epsilon_B)$. The proof relies on the fact that player A knows his own type with certainty and estimates the type of player B correctly with probability $(1-\epsilon)$. Hence as $\epsilon \to 0$ the incentives of player A are very similar to the incentives he has in complete information. In particular the probability ϵ event where he estimates the type of player B incorrectly becomes relatively unimportant.

We slightly abuse notation by defining $u_A(\sigma_A, \sigma_B; \theta_A) := u_A(\Gamma(\sigma_A, \sigma_B); \theta_A)$. Moreover throughout the proof we use the fact that perturbations are restricted: that is to say $s^A_A = \theta_A$ and $s^B_B = \theta_A$.

Proof of 2.4.2 Part (i): $h^\epsilon_B \in H^*_B$

Proof. Suppose this does not hold. Then for some signal $\tilde{s}^B$, and strategy $\tilde{\sigma}_A$ there exists a deviating strategy $\hat{\sigma}_B$ such that:

$$u_B(\tilde{\sigma}_A, h^\epsilon_B(\tilde{s}^B, \tilde{\sigma}_A); \tilde{s}^B_B) < u_B(\tilde{\sigma}_A, \hat{\sigma}_B; \tilde{s}^B_B)$$

Since information perturbations are restricted $s^B_B = \theta_B$, and it follows that:

$$u_B(\tilde{\sigma}_A, h^\epsilon_B(\tilde{s}^B, \tilde{\sigma}_A); \theta_B) < u_B(\tilde{\sigma}_A, \hat{\sigma}_B; \theta_B)$$

[8] Note that this is the part of the proof that does not hold in the setting AFHKT consider. In their setting player B may infer something about his own preferences from the move of player A. In particular, $u_B(\Gamma(\sigma_A, \sigma_B); \theta_B) \neq u_B(\Gamma(\sigma_A, \sigma_B); s^B_B)$.

Consider the following strategy profile:

$$\hat{h}_B^{\epsilon}(s^B, \sigma_A) = \begin{cases} \hat{\sigma}_B & \text{if} \quad (s^B, \sigma_A) = (\tilde{s}^B, \tilde{\sigma}_A) \\ h_B^{\epsilon}(s^B, \sigma_A) & \text{otherwise} \end{cases}$$

Playing according to strategy profile $\hat{h}_B^{\epsilon}$ rather than strategy profile h_B^{ϵ} leads to a higher payoff in the subgame when $(s^B, \sigma_A) = (\tilde{s}^B, \tilde{\sigma}_A)$ and the same payoff otherwise. Hence h_B^{ϵ} cannot be a sequential equilibrium profile of the game (Γ, ν^{ϵ}). This is a contradiction, and completes the proof.

□

Proof of 2.4.2 Part (ii): $h_A^{\epsilon} \in H_A^*(h_B^{\epsilon})$

Proof. First define the following:

$$\begin{aligned} \underline{u} &:= \min_{x, s^A}\{u_A(x; s_A^A)\} \\ \overline{u} &:= \max_{x, s^A}\{u_A(x; s_A^A)\} \\ \sigma_A(s^A) &= h_A^{\epsilon}(s^A, h_B^{\epsilon}) \\ u(s^A) &:= u_A(\sigma_A(s^A), h_B^{\epsilon}(s^A, \sigma_A(s^A)); s_A^A) \\ \hat{u}(s^A) &:= \max_{\tilde{\sigma}_A}\{u_A(\tilde{\sigma}_A, h_B^{\epsilon}(s^A, \tilde{\sigma}_A); s_A^A)\} \end{aligned}$$

We use $\overline{u}$ and $\underline{u}$ to refer to the maximum and minimum payoffs player A could receive, while $u(s^A)$ is the utility player A obtains when he plays according to strategy $\sigma_A(s^A) = h_A(s^A, h_B^{\epsilon})$, player B has the same signal as him ($s^B = s^A$) and plays according to a strategy profile h_B^{ϵ}. Meanwhile $\hat{u}(s^A)$ is the maximum utility player A could obtain in this situation by choosing some arbitrary strategy. Let $\hat{\sigma}_A(s^A)$ be one of these maximizing strategies, so that $\hat{u}(\theta) = u_A(\hat{\sigma}_A(\theta), h_B^{\epsilon}(\theta, \hat{\sigma}_A(\theta)); \theta)$.

Now suppose $h_A^{\epsilon} \notin H_A(h_B^{\epsilon})$. Remembering that h_A is a strategy profile of a sequential equilibrium, we aim for a contradiction. Since $h_A^{\epsilon} \notin H_A(h_B^{\epsilon})$, it follows that for some signal $\tilde{s}^A$ there exists a profitable deviation $\tilde{\sigma}_A$. That is to say:

$$u_A(\sigma_A(\tilde{s}^A), h_B^\epsilon(\tilde{s}^A, \sigma_A(\tilde{s}^A)); \tilde{s}_A^A) < u_A(\tilde{\sigma}_A, h_B^\epsilon(\tilde{s}^A, \tilde{\sigma}_A); \tilde{s}_A^A) \tag{2.1}$$

Using the definition of $\hat{\sigma}_A$, note that the strategy $\hat{\sigma}_A(s^A)$ maximizes the payoff of player A given his signal is s^A. Therefore:

$$u_A(\tilde{\sigma}_A, h_B^\epsilon(\tilde{s}^A, \tilde{\sigma}_A); \tilde{s}_A^A) \leq u_A(\hat{\sigma}_A(\tilde{s}^A), h_B^\epsilon(\tilde{s}^A, \hat{\sigma}_A(\tilde{s}^A)); \tilde{s}_A^A) \tag{2.2}$$

Putting these equations 2.1 and 2.2 together and using the definition of $u(s^A)$ and $\hat{u}(s^A)$ leads to the following:

$$\begin{aligned} u_A(\sigma_A(\tilde{s}^A), h_B^\epsilon(\tilde{s}^A, \sigma_A(\tilde{s}^A)); \tilde{s}_A^A) &< u_A(\hat{\sigma}_A(\tilde{s}^A), h_B^\epsilon(\tilde{s}^A, \hat{\sigma}_A(\tilde{s}^A)); \tilde{s}_A^A) \\ u(\tilde{s}^A) &< \hat{u}(\tilde{s}^A) \end{aligned}$$

Now let $\delta = \hat{u}(\tilde{s}^A) - u(\tilde{s}^A)$ and note that $\delta > 0$. Define an alternative strategy profile $\hat{h}_A^\epsilon$ as follows:

$$\hat{h}_A^\epsilon(s^A) = \begin{cases} \hat{\sigma}_A(s^A) & \text{when} \quad s^A = \tilde{s}^A \\ \sigma_A(s^A) & \text{when} \quad s^A \neq \tilde{s}^A \end{cases}$$

We now show that $\hat{h}_A^\epsilon$ is a profitable deviation. When $s^A \neq \tilde{s}^A$, payoffs under both strategy profiles are equal under both strategy profiles z, so we focus on the case where $s^A = \tilde{s}^A$. Note that in this case $\hat{h}_A^\epsilon(s^A) = \hat{\sigma}_A(\tilde{s}^A)$ and $h_A^\epsilon(s^A, h_B^\epsilon) = \sigma_A(\tilde{s}^A)$. Since information perturbations are restricted, $\theta_A = \tilde{s}_A^A$. Hence it is enough to show that:

$$S \;=\; E_{s^B \in \Theta_B}[u_A(\hat{\sigma}_A(\tilde{s}^A), h_B^\epsilon(s^B, \hat{\sigma}_A(\tilde{s}^A)); \tilde{s}_A^A)] - E_{s^B \in \Theta_B}[u_A(\sigma_A(\tilde{s}^A), h_B^\epsilon(s^B, \sigma_A(\tilde{s}^A)); \tilde{s}_A^A)] > 0$$

First note that with probability $p > (1-\epsilon)$, $s^B = \tilde{s}^A$. In this case the left hand side is equal to $\hat{u}(\tilde{s}^A)$, while the right-hand side is equal to $u(\tilde{s}^A)$. Moreover with probability ϵ any payoff between $u_A \in [\underline{u}, \overline{u}]$ may be obtained. These observations lead to the following bounds:

$$\begin{aligned} E_{s^B \in \Theta_B}[u_A(\hat{\sigma}_A(\tilde{s}^A), h_B^\epsilon(s^B, \hat{\sigma}_A(\tilde{s}^A)); \tilde{s}_A^A)] &\geq (1-\epsilon)\hat{u}(\tilde{s}^A) + \epsilon \underline{u} \\ E_{s^B \in \Theta_B}[u_A(\sigma_A(\tilde{s}^A), h_B^\epsilon(s^B, \sigma_A(\tilde{s}^A)); \tilde{s}_A^A)] &\leq (1-\epsilon)u(\tilde{s}^A) + \epsilon \overline{u} \end{aligned}$$

Using these bounds, the fact that $\delta = \hat{u}(\tilde{s}^A) - u(\tilde{s}^A) > 0$ and assuming $\epsilon < \frac{1}{2}$ gives:

$$\begin{aligned} S &\geq (1-\epsilon)\hat{u}(\tilde{s}^A) + \epsilon \underline{u} - (1-\epsilon)u(\tilde{s}^A) - \epsilon \overline{u} \\ &> \delta - 2\epsilon(\overline{u} - \underline{u}) \end{aligned}$$

$\delta > 0$ and both δ and $(\overline{u} - \underline{u})$ are fixed parameters. Therefore there exists some $\overline{\epsilon}$ such that $S > 0$ whenever $\epsilon \in (0, \overline{\epsilon})$. This shows that $\hat{h}_A^\epsilon$ is a profitable deviation and hence h_A^ϵ cannot be the strategy profile of a sequential equilibrium. This proves the result. □

2.7.2 Example: $F2$ is sufficient but not necessary

Consider again the initial example of the firm and the worker. Now however there is a third type of worker, $(\theta_B = S)$. This worker has an outside option that he prefers to w_H, but otherwise has the same preferences as the high type worker. This outside option can be thought of as another job offer with a high salary. In case he does not reach an agreement with the firm he takes the outside offer. Also suppose that there are two types of firms $(\theta_A \in \{Y, N\})$. One firm would like to hire this special worker by offering him a wage that is even higher than the outside option. The other type of the firm does not want to pay such a high wage.

The references are given in Table 4.

The social choice function where $f(N, L) = f(Y, L) = w_L$, $f(N, H) = f(Y, H) = w_H$, $f(N, S) = d$ and $f(Y, S) = S$ can be implemented in three stages in complete information, where the worker first chooses between the special branch and the normal branch. In case the worker has chosen the special branch, the firm decides

Table 2.4. Example: F2 is not necessary

	Preferences	
Norm fi	$u_P(w_L;\theta) > u_P(w_H;\theta) > u_P(d;\theta) > u_P(S;\theta)$	for $\theta \in \{(N,L),(N,H),(Y,L),(Y,H)\}$
Spec fi	$u_P(w_L;\theta) > u_P(w_H;\theta) > u_P(S;\theta) > u_P(d;\theta)$	for $\theta \in \{Y,S\}$
	$u_P(w_L;\theta) > u_P(w_H;\theta) > u_P(S;\theta) > u_P(d;\theta)$	for $\theta \in \{(N,L),(N,H)\}$
Low t	$u_A(S,\theta) > u_A(w_H;\theta) > u_A(w_L;\theta) > u_A(d;\theta)$	for $\theta_B = L$
High t	$u_A(S,\theta) > u_A(w_H;\theta) > u_A(d;\theta) > u_A(w_L;\theta)$	for $\theta_B = H$
Spec t	$u_A(S,\theta) > u_A(w_H;\theta) > u_A(d;\theta) > u_A(w_L;\theta)$	for $\theta_B = S$

to pay a very high wage S if the worker is indeed the special type and the firm is special, too. It chooses outside option d otherwise. On the other hand, if the worker chooses the normal branch, the game continues as in the basic example.

If the proportion of special firms is sufficiently small and normal workers dislike allocation O sufficiently, then this mechanism is robust to restricted perturbations and the SCF can be implemented robustly, despite requiring three stages. Note however, that this mechanism can be reduced to two stages, when allowing players to move simultaneously in the first stage. Workers report that they are *normal* or *special* and the firm chooses one of S and the default d and one of w_H and w_L. If the worker chooses the special branch the game ends and S or d as chosen by the firm is implemented. If the worker chooses the normal branch then if the firm chose w_H this is implemented. In the final case, where the worker has chosen the normal branch and the firm chose w_L, the worker gets to make a final choice between accepting w_L and rejecting the offer to implement the default d.

2.7.3 Simultaneous moves

We now provide an example to show that the credible threat condition is not necessary for robust implementation under restricted information perturbations when allowing players to move simultaneously.

Consider the case where there are two players A and B. For simplicity assume that the preferences of player B are fixed, while player A's preferences are given by θ or $\hat{\theta}$. We assume that player A knows his preferences with certainty while the signal player B receives is equal to player A's preferences with probability $1-\epsilon$ and equal to the other preference with the remaining probability ϵ. Now consider the following mechanism:

		B	
		θ	$\hat{\theta}$
A	θ	Γ	$(0,0,0)$
	$\hat{\theta}$	$\theta:(1,0,10)$ $\hat{\theta}:(0,1,0)$	$(7,7,3)$

Figure 2.4. Simultaneous moves

In the first stage of the game both players simultaneously choose between reporting θ and reporting $\hat{\theta}$. This is described in Figure 2.4. If both players report $\hat{\theta}$ then the game ends and players receive the payoffs given in brackets. The first number corresponds to the payoff of player A if he is type θ, the second number is the payoff of player A if he is type $\hat{\theta}$ and the third number is the payoff of player B. Similarly if player B reports $\hat{\theta}$ and player A reports θ, the payoffs are $(0,0,0)$ and the game ends.

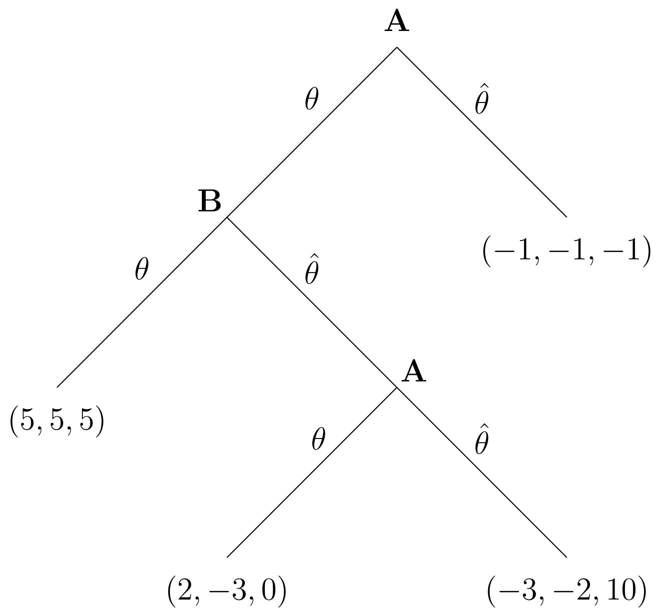


Figure 2.5. Mechanism Γ

Now consider the case where player A reports $\hat{\theta}$ and player B reports θ. In this case player A has got a second move and chooses again between the reports θ and $\hat{\theta}$ which correspond to payoff vectors of $(1,0,10)$ and $(0,1,0)$ respectively.

In the case where both players report θ, they start playing the mechanism Γ given by the game tree in Figure 2.5 in the second stage.

It can easily be checked that the underlying preferences do not satisfy the credible threat condition.

We now show that despite this fact, the simultaneous move mechanism described above robustly implements the social choice function with payoffs $(5,5)$ in state θ and $(7,3)$ in state $\hat{\theta}$ under restricted information perturbations. For simplicity we assume that the states θ and $\hat{\theta}$ are ex-ante equally likely.

First note that the unique equilibria under complete information are given by the reports $(\theta,\theta,\theta,\theta)$ in state θ and $(\hat{\theta},\hat{\theta})$ in state $\hat{\theta}$. Hence the desired SCF is implemented under complete information.

Now consider the case where player A's realised preferences are θ. If the mechanism

Γ is reached, player A has got a dominant strategy to re-report his preferences as θ. Moreover whenever player A's preferences are θ his initial report is θ. This ensures him a payoff of 2 which is greater than any payoff he can hope to achieve by reporting $\hat{\theta}$, since the reports $(\hat{\theta}, \hat{\theta})$ are not an equilibrium. Knowing this, player B assigns a high probability to player A's preferences being θ whenever he observes A re-reporting himself as θ and mechanism Γ is played. As a consequence B also reports θ and the desired allocation is implemented. There cannot be a case, where player A re-reports his preferences as θ and player B then assigns a higher probability to A's preferences being $\hat{\theta}$ than before the first stage.

Secondly consider the case where player A's preferences are given by $\hat{\theta}$. Then the reports $(\hat{\theta}, \hat{\theta})$ are an equilibrium. Player A cannot gain by deviating as there does not exist an allocation which gives him a higher payoff. Player B cannot gain by deviating to another report either: If he reports θ player A has got another move where he has a dominant strategy to re-report $\hat{\theta}$, leaving player B with a payoff $0 < 3$. Hence the report $(\hat{\theta}, \hat{\theta})$ is an equilibrium if the state is $\hat{\theta}$.

Note also that it is the only equilibrium in this state. In particular the mechanism Γ played when the reports are (θ, θ) cannot be an equilibrium, as it would implement an allocation $(-1, -1)$, which neither of the players likes.

2.7.4 Virtual Implementation

We now prove that the SPE equilibrium in mixed strategies stated in section 5.1 is indeed the unique equilibrium of the mechanism described and hence virtually implements the desired SCF.

Proof. Let $\delta = \sqrt{\epsilon}$ and suppose ϵ is sufficiently small. In this case, if more than fraction δ of low types choose the left branch, the principal - on observing signal s^L will challenge the report. This is because the report is sufficiently likely to originate from a low type and hence:

$$u_P(w_L) < \frac{\delta(1-\epsilon)m_L}{\delta(1-\epsilon)m_L + \epsilon m_H} u_P(y_L) + \frac{\epsilon m_H}{\delta(1-\epsilon)m_L + \epsilon m_H} u_P(y_H)$$

Secondly note that if more than fraction δ of low types choose the right branch, the principal - on observing signal s^L will accept the report. This is because the report is sufficiently likely to originate from a low type and hence:

$$u_P(w_H) > \frac{\delta(1-\epsilon)m_L}{\delta(1-\epsilon)m_L + \epsilon m_H} u_P(x_L) + \frac{\epsilon m_H}{\delta(1-\epsilon)m_L + \epsilon m_H} u_P(x_H)$$

Suppose there is a SPE where more than δ low types choose the left branch in the first round. Then these low types with probability greater than $(1-\epsilon)$ would receive payoff $u_L(y_L)$. If ϵ is sufficiently low, it is optimal for these low types to deviate and choose the right branch in the first round guaranteeing a payoff higher than $u_L(y_L)$. It follows that in any SPE a fraction at least $(1-\delta)$ low types chooses the right branch in the first stage.

Since a high fraction of low types report L in the first round, it follows from above that the principal - on observing a report of L and signal s^L - will always accept the report and implement w_H. Since this is the highest payoff a low type can receive it follows that all low types will report L in the first round.

Since only high types choose the left branch, it follows that the firm will accept to pay w_L, whenever a worker chooses the left branch in the first stage. Therefore high type workers have a choice between (i) choosing the left branch and receiving a guaranteed payoff of $u_H(w_L)$ and (ii) choosing the right branch. Suppose all high types choose the right branch. Then the firm - on observing a worker has chosen the right branch and a signal s^H - will challenge the worker by moving to the third stage - and x_H will be implemented. In this case high types - preferring w_H to x_H - would have an incentive to deviate and choose the left branch initially. Suppose now on the other hand that all high types choose the left branch. Then the firm - on observing that the right branch has been chosen and a signal s^H - will not challenge and w_H will be implemented. In this case high types - preferring w_H to w_L - would have an incentive to deviate.

It follows from the two observations above that high types must mix in the first stage. Moreover for high types to be indifferent over their mixing, it follows that the principal must mix in the second stage after observing a report L and a signal s^H. The mixing parameters α and β are calculated above, and hence this is the unique SPE.

$\square$

Chapter 3

Generalised Weighted Raiffa Solutions

3.1 Introduction

In the 1950s Nash (1950) and Raiffa (1953) independently introduced two solution concepts to a general bargaining problem, which has since been studied extensively. While Nash motivated his solution by appealing to the independence of irrelevant alternatives (IIA) axiom, Raiffa focused on the interim steps necessary to reach a final agreement. Both of these original solutions require that identical players are treated equally, and effectively assume that all players have equal bargaining weight. However in many situations this is not the case, and one party can influence the outcome of negotiations more effectively than another due to different levels of skill or commitment.

Acknowledging this fact, Harsanyi and Selten (1972) introduced a family of weighted Nash solutions, where each player is assigned a bargaining weight. These bargaining weights concisely model differences between players, which are not related to the players' payoff functions. Firstly one can think of the bargaining weights reflecting player specific characteristics: a player who is more patient or who can quickly respond with a counter-proposal might be expected to achieve a higher allocation than a player who is less patient or takes longer to respond to proposals. Secondly differences in bargaining weight may reflect differences in the negotiation procedure or institutional framework. For instance permanent members of the UN security council have a veto while other countries do not, and this system means that permanent members are in a stronger position to negotiate favourable agreements than non-permanent members. This paper extends the original solution proposed by Raiffa (1953) to accommodate the case where players have asymmetric bargaining weights.

Weighted versions of other bargaining solutions such as the egalitarian solution (Thomson (1994)) and Kalai-Smorodinksy solution (Kalai (1977), Thomson (1994), Dubra (2001)) have also been proposed. Although the Raiffa solutions have received considerable attention (see for example Anbarci and Sun (2013) and Trockel (2015)) - as far as we are aware - these solutions have not been generalised to accommodate unequal bargaining weights. In this paper we introduce and provide cooperative and non-cooperative foundations for a family of weighted Raiffa solutions. The cooperative foundation appeals to two of the original axioms proposed by Nash and a monotonicity axiom focusing on interim agreements. Meanwhile the non-cooperative foundation shows that these solutions can be implemented using simple bargaining models where offers are made either intermittently or where the identity of the proposer is persistent.

Weighted bargaining solutions are used in many economic applications. Prominent examples include wage bargaining in labour economics (see for example Shimer (2005))) and bankruptcy negotiations in the finance literature (see Yue (2010)). The vast majority of these applications use the weighted Nash solution, due to its strong cooperative and non-cooperative foundations. We show that weighted Raiffa solutions have similarly strong foundations and hence should be considered as an alternative. In particular considering these different solutions alongside each other could serve as a robustness check, and help determine whether the predictions of a model are sensitive to the solution concept used. Furthermore our results underline the fact that bargaining models with patient players and relatively close deadlines are associated with the Raiffa solution, while bargaining models with relatively impatient players and distant deadlines are associated with the Nash solution. This has implications for the design of negotiation protocols, since policy makers may be able to affect the outcome of negotiations simply by changing the timing of the deadline.

The second section outlines a characterisation for the family of weighted Raiffa solutions. It is related to the characterisation provided by Diskin et al. (2011) who introduce and characterise a family of p-Raiffa solutions. This family includes the discrete and continuous Raiffa solutions introduced by Raiffa (1953) as the extreme cases when $p = 1$ and as p approaches 0 respectively. We build on this approach with two important differences. Firstly, we do not use a symmetry axiom in order to characterise a family of weighted (λ, p)-Raiffa solutions. Secondly, rather than axiomatizing on the sequence of interim agreements, our characterisation uses weaker axioms related to the eventual bargaining solution wherever possible. These weaker axioms are commonly used in bargaining theory, making it easier to compare with existing results.

In order to be able to make further comparisons we also provide a new axiomatization for weighted Kalai-Smorodinsky solutions which uses similar axioms. This helps to show how the *weighted Raiffa solution*, the *weighted Kalai-Smorodinsky solution* and the *weighted egalitarian solution* can all be axiomatized by appealing to different versions of a monotonicity axiom combined with scale invariance and Pareto optimality.

The third section provides non-cooperative foundations for weighted Raiffa solutions. We introduce a class of bargaining models that can be used to approximately implement any weighted Raiffa solution. The games considered have a finite number of rounds, where players do not discount and the identity of the proposer is determined by a Markov process with $(n + 1)$ states. In state i player i makes a proposal, while in state $(n+1)$ no offer is made. If an offer is accepted by all other players then it is implemented. Otherwise negotiations continue to the next round. In particular we show that specific types of this general model - when offers are intermittent or the identity of the proposer is persistent - implement a weighted Raiffa solution.

This class of bargaining models generalises the finite horizon models studied by Stahl (1972) and Sjostrom (1991). Moreover it is also related to the infinite horizon model considered by Binmore et al. (1986) which implements the Nash solution. In particular our model has a strong resemblance with the generalised version of this model considered by Britz et al. (2010). While they consider an infinite horizon model with a discount factor where the identity of the proposer is determined by a Markov process, we consider a similar environment with a finite horizon and no discount factor. This shows that when discounting is the dominant factor a weighted Nash solution is implemented, while when deadlines are the dominant factor a weighted Raiffa solution is implemented. Hence our analysis extends the results in Gomes et al. (1999) and Imai and Salonen (2012), who study this effect in settings where all players are equally likely to be selected as proposer in each round. We now outline the bargaining problem and introduce the family of Raiffa solutions.

3.1.1 The bargaining problem

Consider n-player bargaining problems where the set of players is denoted by $N = \{1, , \ldots, n\}$. Players negotiate over how to split a cake of size one with free disposal. The default allocation is normalised to $\mathbf{0} \in \mathrm{Re}^n$, while the set of feasible allocations $X \subset \mathrm{Re}^n$ is defined as follows:

$$X = \left\{ \mathbf{x} : \sum x_i \leq 1 \text{ and } x_j \geq 0 \text{ for all } j \in N \right\}$$

Each player i has a utility function $u_i : [0, 1] \mapsto \mathrm{Re}$, which maps a quantity x_i to a payoff s_i. We assume that all u_i are strictly increasing and strictly concave.[1] Using these utility functions, the set of feasible payoff vectors $S \subset \mathrm{Re}^n$ is defined as follows:

$$S = \left\{ \mathbf{s} : \text{ there exists } \mathbf{x} \in X \text{ such that } s_i = u_i(x_i) \text{ for all } i \in N \right\}$$

Since the utility functions are concave and X is a simplex, it follows that S is convex. Define the default allocation of player i to be $d_i = u_i(0)$ and let $\mathbf{d} = (d_1, d_2, ..., d_n)$ be the vector of all players' default allocations. We refer to the pair

[1] This models situations where all players strictly prefer more cake to less cake and are strictly risk averse.

$(S, \mathbf{d})$ as the utility representation of a bargaining problem. An n-player bargaining solution Φ maps a utility representation $(S, \mathbf{d})$ to a solution $\Phi(S, \mathbf{d}) \in S$. This final requirement captures the fact that a bargaining solution Φ selects a feasible payoff vector. When $\mathbf{s} \in S$, we define the ideal point for player i as $m_i(\mathbf{s}|S)$. This is the highest utility level that player i could receive while still ensuring that each player j receives a utility of at least s_j.

$$m_i(\mathbf{s}|S) = \max\{\hat{s}_i | \hat{\mathbf{s}} \in S \text{ and } \hat{s}_j \geq s_j \text{ for all } j \in N\}$$

We define $m(\mathbf{s}|S)$ to be the vector of such utility levels, and refer to this as the *ideal point given* $\boldsymbol{s}$. Note that by convexity of S, $m_i(\mathbf{s}|S)$ is strictly decreasing in s_j whenever $j \neq i$. This captures the fact that players are in a competitive situation: if the utility allocation s_j of an opponent $j \neq i$ increases, keeping that of the remaining players at least constant, then the highest feasible utility of player i - namely $m_i(\mathbf{s}|S)$ is reduced.

3.1.2 The family of Raiffa solutions:

Ideal points have been used by Raiffa (1953) and Diskin et al. (2011) to define bargaining solutions as a limit of an iterative process, and below we extend this approach. The initial point of the iteration is fixed to be the default allocation $m_i(\mathbf{s}(k)|S)$. Subsequent steps $\mathbf{s}(k+1)$ are determined by the current disagreement point $\mathbf{s}(k)$ and the current ideal point $m(\mathbf{s}(k)|S)$. This captures the fact that during a negotiation some player i considers two pieces of information: first he focuses on what he is sure to obtain, namely the current disagreement point $\mathbf{s}_i(k)$; secondly he focuses on what he could possibly obtain in an ideal world namely $\mathbf{s}_i(k)$. Finally the bargaining solution $\Phi(S, \mathbf{d})$ is given to be the limit of this process. Each step in the iterative process can be thought of as an interim agreement, where every player prefers the interim agreement $\mathbf{s}(k+1)$ to the interim agreement $\mathbf{s}(k)$.

The two-player discrete Raiffa solution was the first iterative bargaining solution to be introduced in Raiffa (1953). Here the new agreement $\mathbf{s}(k+1)$ is calculated by taking the midpoint between the current agreement $s(k)$ and the current ideal point $m(S, s(k))$. By convexity of S this point lies in S.

Definition 18 (Discrete Raiffa solution)**.**
The two-player discrete Raiffa solution is defined as $\Phi(S, \boldsymbol{d}) = \lim_{k\to\infty} \boldsymbol{s}(k)$ *where:*

- $\boldsymbol{s}(0) = \boldsymbol{d}$
- $\boldsymbol{s}(k+1) = \frac{1}{2}\boldsymbol{s}(k) + \frac{1}{2}m(\boldsymbol{s}(k), S)$

This solution is illustrated in Figure 3.1.

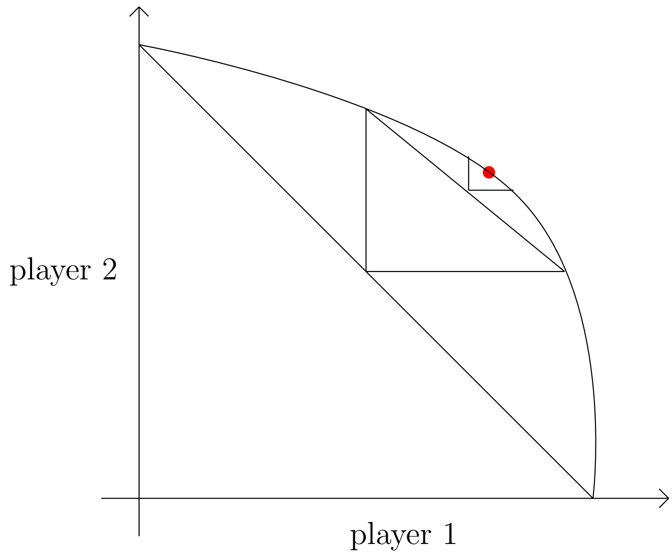


Figure 3.1. Discrete Raiffa solution

Salonen (1988) extends this solution to n-players, and provides an axiomatization for these n-player discrete Raiffa solutions. While in the two-player case each player moves half way towards his current ideal point, for the n-player case this may lead to payoffs which are not feasible. Therefore in the n-player case each player moves a fraction $\frac{1}{n}$ towards their ideal point.

A more general set of n-player Raiffa solutions is considered by Diskin et al. (2011). The Raiffa solutions in this family are characterized by a parameter p determining the step size of interim agreements and hence the speed of convergence to a solution. When $p = 1$, the p-Raiffa solution corresponds to the discrete Raiffa solution. However when $p < 1$ interim agreements lie closer together, and the sequence $\mathbf{s}(k)$ converges more slowly. As $p \to 0$, the interim agreements become arbitrarily close.

This approximates the n-player continuous Raiffa solution. The two player version of this solution was introduced by Raiffa (1953) and axiomatized by Peters and Damme (1991). The family of solutions considered by Diskin et al. (2011) can be stated as follows:

Definition 19 (p-Raiffa solution)**.**
For $p \in (0, 1]$ *the* n*-player* p*-Raiffa solution is defined as* $\Phi^p(S, \boldsymbol{d}) = \lim_{k\to\infty} \boldsymbol{s}(k)$ *where:*

- $\boldsymbol{s}(0) = d$
- $\boldsymbol{s}(k+1) = \left(1 - \frac{p}{n}\right)\boldsymbol{s}(k) + \frac{p}{n} m(S, \boldsymbol{s}(k))$

For the two-player case with bargaining weights equal to 0.7 and 0.3 respectively, this is illustrated in Figure 3.2.

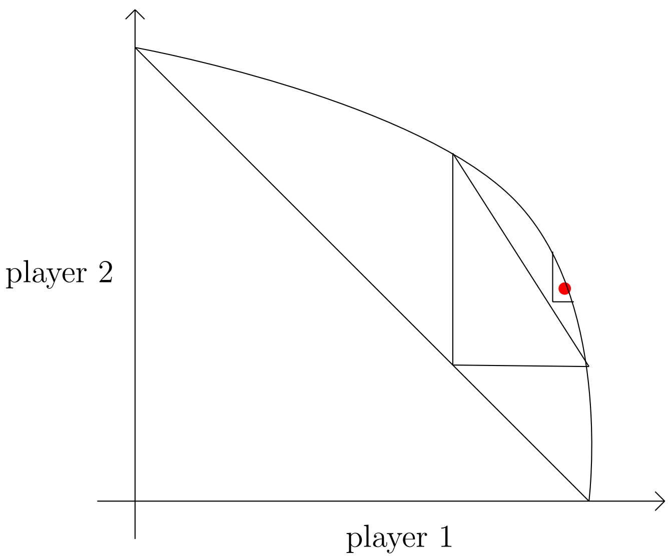


Figure 3.2. P-Weighted Raiffa solution P =(0.7,0.3)

Our contribution is to generalise this set of Raiffa solutions to a more general set of weighted bargaining solutions where players may have different bargaining weights. A bargaining weight $\lambda_i \in (0, 1)$ is assigned to each player, and the parameter $\lambda = (\lambda_1, \lambda_2..., \lambda_n)$ denotes the vector of players' exogenous bargaining weights. Without loss of generality we assume that $\Sigma\lambda_i = 1$, and define (λ, p)-Raiffa solutions as follows:

Definition 20 ((λ, p)-Raiffa solution)**.**
The n-player (λ, p)-Raiffa solution is defined as $\Phi^{\lambda,p}(S, \boldsymbol{d}) = \lim_{k\to\infty} s(k)$ *where:*

- $\boldsymbol{s}(0) = \boldsymbol{d}$
- $\boldsymbol{s}_i(k+1) = \Big(1 - p\lambda_i\Big)\boldsymbol{s}(k) + p\lambda_i m(\boldsymbol{s}(k)|S)$

For the two player case, where $\lambda = (0.75, 0.25)$ and $p = 0.8$, this solution is illustrated in Figure 3.3.

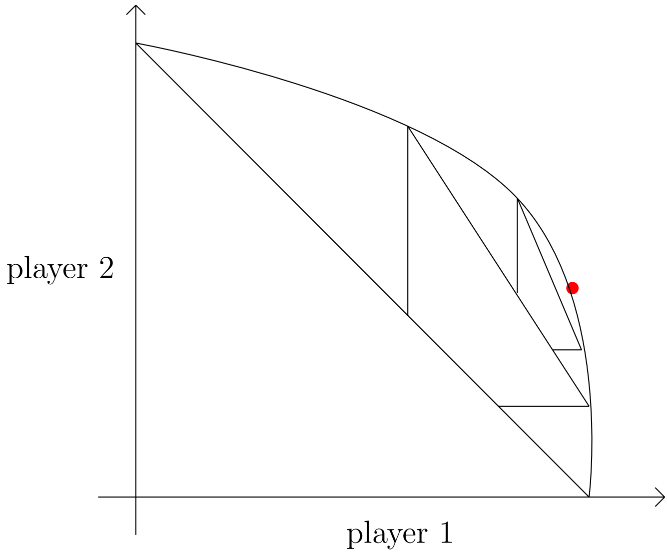


Figure 3.3. Weighted Generalised Raiffa solution: (0.75, 0.25, 0.2)

The next two sections provide a cooperative and a non-cooperative foundation for this family of bargaining solutions.

3.2 Cooperative foundation

In this section we provide a cooperative foundation for (λ, p)-Raiffa solutions. First we introduce some additional notation. The set of individually rational points that all players prefer over the default is referred to as $S_{\mathbf{d}} := \{\mathbf{s} \in S | s_i \geq d_i \text{ for all } i\}$. We use $PF(S) := \{\mathbf{v} \in S | m(\mathbf{v}|S) = \mathbf{v}\}$ to refer to those points in S that are Pareto optimal. It is further assumed that any point which is weakly Pareto optimal is also strictly Pareto optimal - ie $m_i(\mathbf{v}|S) = \mathbf{v}_i$ implies $\mathbf{v} \in PF(S)$.[2]

Unlike other solution concepts, the Raiffa solution explicitly specifies a sequence of interim points between the default allocation and the bargaining solution. These interim points can be thought of as a series of interim agreements that players make before reaching the final solution. To capture this idea that a bargaining solution may be reached through a number of interim agreements, Diskin et al. (2011) model interim solutions by using a step function. In a similar way we say that an interim solution maps a bargaining problem $(S, \mathbf{s})$ to a unique point $\delta(S, \mathbf{s}) \in S$ whenever $\mathbf{s} \in S$. An interim solution $\delta(S, \mathbf{d})$ can be interpreted as a first interim agreement that players reach as they move towards the eventual bargaining solution $\Phi(S, \mathbf{d})$. We say that δ is associated with the bargaining solution Φ if repeated applications of the interim solution δ eventually approximate the bargaining solution Φ:

Definition 21 (Interim solutions δ)**.**
An interim solution δ is associated with a bargaining solution Φ iff $\Phi(S, \boldsymbol{d}) = \lim_{k\to\infty} \boldsymbol{d}^k$ where $\boldsymbol{d}^0 = \boldsymbol{d}$ and $\boldsymbol{d}^{k+1} = \delta(S, \boldsymbol{d}^k)$ for all $(S, \boldsymbol{d})$

Note that the interim solution $\delta = \Phi$ is trivially associated with the bargaining solution Φ. Hence it is clear that any bargaining solution Φ is associated with at least one interim solution. We say an interim solution δ is non-trivial if $\mathbf{d} \notin PF(S)$ implies $\delta(S, \mathbf{d}) \notin PF(S)$. Following most of the literature we focus only on Pareto optimal and scale-invariant bargaining solutions. Furthermore we require that any points which are not individually rational do not affect the final bargaining solution. This leads to the following axioms:

Scale invariance (SI)
If $\mathbf{b}, \mathbf{c} \in \mathrm{Re}^n_+$ and $F(\mathbf{x}) = \mathbf{b} \cdot \mathbf{x} + \mathbf{c}$, then $\Phi(F(S), F(d)) = F(\Phi(S, d))$

Pareto optimality (PO)
If $s'_i > \Phi_i(S, d)$ for all $i \in N$, then $\mathbf{s}' \notin S$

Irrelevance of non-individually rational points (IIR)

$$\Phi(S, \mathbf{d}) = \Phi(T, \mathbf{d}) \quad \text{whenever} \quad S_{\mathbf{d}} = T_{\mathbf{d}}$$

[2]Note that this follows immediately if the bargaining set S is associated with strictly increasing utility functions

Along with these standard axioms, we introduce an axiom requiring that if any feasible allocation $s \in S_{\mathbf{d}}$ remains feasible when the bargaining set changes from S to T, then such a change is weakly beneficial for all players. Therefore if additional allocations become feasible, then no player becomes worse off. This requirement is captured as follows:

Φ-monotonicity (Φ-MON)

$$\text{If} \quad S \subseteq T, \quad \text{then} \quad \Phi_i(S, \mathbf{d}) \leq \Phi_i(T, \mathbf{d})$$

We say that moving from S to T is an *enlargement of the feasible set* if and only if $S \subseteq T$. Hence the Φ monotonicity condition says that if the feasible set is enlarged, then no player becomes worse off. Although this axiom seems plausible, it is not compatible with Pareto optimality and scale invariance. This is shown by the following result due to Thomson (1994):

Proposition 3.2.1 ((λ)-Egalitarian solutions)**.**
If Φ satisfies (IIR), (PO) and (Φ-MON) then Φ is a (λ)-weighted Egalitarian solution. The (λ)-weighted Egalitarian solution is not scale invariant and is defined uniquely as follows:

$$EG_{\lambda}(S, \mathbf{d}) \in \Big\{ \mathbf{s} | s_i = d_i + k\lambda_i \;\; \textit{for some} \;\; k \in \mathrm{Re}_+ \Big\} \cap PF(S)$$

This shows that the axiom of Φ-monotonicity is too strong to characterise scale-invariant and Pareto optimal solutions. In light of this result, we consider the following weaker axiom of δ-monotonicity:

δ-monotonicity (δ-MON) For some interim solution δ associated with Φ:

$$\text{If} \quad S \subseteq T, \quad \text{then} \quad \delta_i(S, \mathbf{d}) \leq \delta_i(T, \mathbf{d}) \text{ for all } i$$

The δ-monotonicity axiom also appeals to the idea that enlarging the feasible set should benefit players. However while Φ monotonicity requires that *an enlargement of the feasible set* weakly increases the final allocation of any player, δ monotonicity only requires that for some step function δ associated with Φ *an enlargement of* *the feasible set* weakly increases the allocation of any player after the first interim agreement. More precisely there exists a δ associated with Φ such that if $S \subseteq T$, then the payoff $\delta_i(S, \mathbf{d})$ assigned to player i after the first interim agreement when the bargaining set is S is weakly less than the payoff $\delta_i(T, \mathbf{d})$ assigned to player i after the first interim agreement when the bargaining set is T. Using this axiom, we now state the main result of this section:

Proposition 3.2.2 ((λ, p)-Raiffa solutions - monotonicity)**.**
If Φ satisfies (IIR), (SI), (PO), (δ-MON) then Φ is a (λ, p)-weighted Raiffa solution.

The result shows that weighted Raiffa solutions are the only bargaining solutions that satisfy a monotonicity condition on a path of interim agreements. First note that the weighted Raiffa solutions satisfy the axioms: clearly (IIR), (SI) and (PO) are satisfied. To see (δ-MON) is also satisfied consider the following interim solution:

$$\delta_i(S, \mathbf{d}) = p\lambda_i m_i(\mathbf{d}|S) + (1 - p\lambda_i)\mathbf{s}_i \quad \text{for all} \quad i$$

Showing that no other solutions satisfy these axioms is non-trivial. The proof first considers a weaker axiom, which says that if the maximum gain available to player i - namely $m_i(\mathbf{d}|S)$ - increases as the bargaining set changes from S to T, then the allocation player i will receive after the first interim agreement will also increase:

δ-initial gain (δ-INITIAL) For some interim solution δ associated with Φ:

$$\text{If} \;\; m_i(\mathbf{d}|S) \leq m_i(\mathbf{d}|T) \;\; \text{then} \;\; \delta_i(S, \mathbf{d}) \leq \delta_i(T, \mathbf{d}) \text{ for all } \mathbf{d} \in S$$

Since $S \subset T$ implies $m_i(\mathbf{d}|S) \leq m_i(\mathbf{d}|T)$ it follows that δ-initial gain is implied by δ-monotonicity. Hence to prove the previous proposition, it is sufficient to prove the following lemma:

Lemma 3.2.3 ((λ, p)-Raiffa solutions - initial gain)**.**
If Φ satisfies (SI), (PO), (δ-INITIAL) then Φ is a (λ, p)-weighted Raiffa solution.

Since Raiffa solutions - unlike other bargaining solutions - determine interim agreements solely using the current interim agreement which can be interpreted as the current default allocation $\mathbf{d}$ and the current ideal point $m(\mathbf{d}|S)$ this result seems intuitively plausible. The proof builds on techniques from Diskin et al. (2011), who axiomatize the symmetric version. However it is by no means a simple extension, and additional complications arise here for two reasons: first dropping the symmetry axiom allows a wider class of weighted bargaining solutions to be axiomatized; secondly the Φ-scale invariance axiom here is weaker than the δ-scale invariance axiom used by Diskin et al. (2011).[3] This complicates the proof, but makes comparisons with other axiomatizations easier and clarifies the exact role of the interim agreement function. The next section explores such a comparison between this family of Raiffa solutions and the Kalai-Smorodinsky solution.

3.2.1 Weighted Kalai-Smorodinsky solutions

In this section we provide a new axiomatization for the family of weighted Kalai-Smorodinsky (KS) solutions. Unlike existing results - such as that found in Dubra (2001) - the axiomatization below appeals only to the concept of monotonicity. These solutions are defined as follows:

Definition 22. *Given a bargaining problem $(S, \boldsymbol{d})$ and bargaining weights λ the weighted KS solution is uniquely defined as follows:*

$$KS_\lambda(S, \boldsymbol{d}) = \left\{ \boldsymbol{s} | s_i = d_i + a\lambda_i \Big(m_i(S, \boldsymbol{d}) - d_i \Big) \ \text{ for all } i \cap WP(S) \ \text{ where } a \in \mathrm{Re}_+ \right\}$$

In order to motivate the following two axioms, we introduce the concept of *relative bargaining strength.* Given a bargaining problem $(S, \mathbf{d})$ we make the following definition:

$$R_{i,j}(S, \mathbf{d}) = \frac{m_i(S, \mathbf{d}) - d_i}{m_j(S, \mathbf{d}) - d_j}$$

For any bargaining problem $(S, \mathbf{d})$ the value $R_{i,j}(S, \mathbf{d})$ is referred to as the *relative bargaining strength* of player i to player j. This value is given by the ratio between the maximum gains player i can hope to achieve to the maximum gains player j can hope to achieve. We say that a change in the bargaining set from S to T

[3] In particular the δ-invariance used by Diskin et al. (2011) immediately implies Φ-invariance, while Φ-invariance does not imply δ-invariance.

places player i in a relatively stronger bargaining position compared with another player j if $R_{i,j}(S, \mathbf{d}) \leq R_{i,j}(T, \mathbf{d})$. An *enlargement of the bargaining set* also puts player i in a *relatively stronger bargaining position compared to another player j*, if the maximum gains player i can hope to achieve has increased proportionally more compared to the maximum gains player j can hope to achieve.

This intuition motivates another way of restricting that the Φ-monotonicity axiom. The alternative restriction says that if the *initial bargaining strength* of player i weakly increases then either a player's final allocation weakly increases or the relative bargaining strength of player i compared to some other player j strictly decreases. This leads to the following axiom:

Φ- initial monotonicity (Φ-R-INITIAL)
Suppose $m_i(\mathbf{d}|S) \leq m_i(\mathbf{d}|T)$. Then:

$$\text{Either} \quad \Phi_i(S, \mathbf{d}) \leq \Phi_i(T, \mathbf{d}) \quad \text{or} \quad R_{i,j}(S, \mathbf{d}) > R_{i,j}(T, \mathbf{d}) \quad \text{for some} \quad j$$

Combined with Pareto optimality, scale invariance and symmetry this axiom uniquely characterises the KS-solution. Moreover it is strictly weaker than the monotonicity axiom introduced by Kalai (1977).[4] However dropping the symmetry axiom does not lead to a unique family of weighted bargaining solutions: indeed the large class of monotonic solutions that satisfy the remaining three axioms has been fully characterised by Peters and Damme (1991). Therefore a characterisation of the family of KS weighted solutions requires an additional axiom.

The new axiom we propose uses a similar restriction to the one imposed on the Φ-monotonicity axiom above, but the restriction is placed on the weaker δ-monotonicity axiom. Although the δ-monotonicity axiom is attractive - the same interim agreement $\delta(S, \mathbf{d})$ may not always be appropriate for two bargaining problems $(S, \mathbf{d})$ and $(T, \mathbf{d})$ where players have the same *initial bargaining strength*. In particular the same interim agreement may not be appropriate in those cases where

[4]The monotonicity axiom introduced by Kalai (1977) is slightly stronger. Using the notation above, the first clause becomes $S \subset T$ and the second clause becomes: Either $\Phi_i(S, \mathbf{d}) \leq \Phi_j(S, \mathbf{d})$ or $m_j(S, \mathbf{d}) < m_j(T, \mathbf{d})$. We use the slightly weaker axiom above in order to show a connection with the new axiom introduced below

players initially have the same *initial bargaining strength* but when the agreement $\delta(S, \mathbf{d})$ leads to one player having much greater *relative bargaining strength* under the new problem $(S, \delta(S, \mathbf{d}))$ compared to the new problem $(T, \delta(S, \mathbf{d}))$. In such cases the player who is moving to a position with less *relative bargaining strength* may demand a higher allocation, to compensate for his loss in *relative bargaining power*. This motivates the following axiom:

δ-restricted monotonicity (δ-R-MON)
Suppose $m_i(S, \mathbf{d}) \leq m_i(T, \mathbf{d})$. Then there exists a non-trivial δ associated with Φ such that:

$$\text{Either} \quad \delta_i(S, \mathbf{d}) \leq \delta_i(T, \mathbf{d}) \quad \text{or} \quad R_{i,j}(S, \delta(S, \mathbf{d})) < R_{i,j}(T, \delta(S, \mathbf{d})) \quad \text{for some} \quad j \neq i$$

This axiom says that if the *initial bargaining strength* of player i (weakly) improves after a change from S to T then either this player will have a weakly higher allocation $\delta_i(T, \mathbf{d})$ after an interim agreement or the relative bargaining strength of player i compared to player j improves when the default is $\delta(S, \mathbf{d})$ and the bargaining set changes from S to T. This additional requirement allows a player to receive less after the first interim agreement in those cases when his *initial bargaining strength* has improved but where the same interim agreement $\delta(S, \mathbf{d})$ would result in a significantly greater *relative bargaining strength.* This extra restriction models the fact that players may take into account their future relative bargaining strength when deciding upon interim agreements: a player may be more reluctant to make an agreement if it leads to a situation where he is in a relatively weak bargaining position.

Since δ-restricted monotonicity is weaker than δ-monotonicity, it follows from above that weighted Raiffa solutions satisfy this axiom. It is shown in the appendix that weighted Kalai-Smorodinsky solutions also satisfy this axiom. Using the Φ-restricted monotonicity axiom above leads to the following characterisation:

Proposition 3.2.4 (Weighted Kalai-Smorodinsky solution)**.** *If Φ satisfies (SI), (PO), (Φ-RMON) and (δ-RMON), then Φ is a λ-weighted Kalai-Smorodinsky solution.*

Table 3.1. Axiom summary

	Generalized Raiffa	Kalai-Smorodinsky	Egalitarian	Nash
Pareto Optimality	✔	✔	✔	✔
Scale Invariance	✔	✔		✔
δ-monotonicity	✔		(✔)	
Restricted δ-monotonicity	(✔)	4^{*}	(✔)	
Φ-monotonicity			✔	
Restricted Φ-monotonicity		✔	(✔)	
IIA			(✔)	✔
Restricted IIA		4^{**}	(✔)	(✔)

Another axiomatization of the weighted family of KS solutions is provided by Dubra (2001). This axiomatization is based on Φ-monotonicity and a weakened version of the irrelevant alternatives axiom (IIA) used by Nash. In contrast here we show that weighted KS solutions can also be characterised by using a weakened version of δ-monotonicity rather than a weakened version of (IIA). A full summary of these results is given in Table 1. The unbracketed entries are sufficient to characterise the relevant family of weighted solutions. Meanwhile the bracketed entries are not needed for the characterisation, but are further properties satisfied by the solution in question. The starred entries capture the two alternative ways to axiomatize the weighted Kalai-Somorodinsky solution:

This shows the strong connection between the cooperative foundations of weighted Raiffa solutions and weighted KS solutions. Both solution families can be characterised by appealing to the concepts of monotonicity, scale invariance and Pareto optimality. On the one hand, weighted KS solutions are more forward-looking and focus primarily on the eventual outcome. This family can can be characterised using restricted monotonicity axioms on both the bargaining solution as well as the path of interim solutions. In contrast the weighted Raiffa solutions put more focus on the current default point, and can be characterised by a stronger monotonicity axiom on the path of interim solutions.

3.3 Non-cooperative foundation

We now consider non-cooperative foundations for (λ, p)-Raiffa solutions, by showing how these solutions can arise from simple non-cooperative games. More precisely we suggest bargaining procedures that could be used by a planner to implement a certain (λ, p)-Raiffa solution, in situations where the set of utility functions $(u_i)_{i \in N}$ is common knowledge among players but is not known by the planner. The procedures we consider implement bargaining solutions to any arbitrary degree of accuracy. Exact implementation can be achieved by modifying the procedure considered by Trockel (2011).

First we outline the bargaining procedure. Without loss of generality, consider bargaining sets S where the default is normalized to $\mathbf{d} = \mathbf{0}$. Players have T rounds to reach an agreement. In each round the bargaining procedure may be in one of $(n+1)$ states. When the procedure is in state i, player i makes an offer, while if the process is in state $(n+1)$ no offer is made. The player selected to make an offer proposes a feasible allocation. If all other players accept the game ends and the allocation proposed is implemented. Otherwise the game continues to the next round. If after T rounds no agreement is reached, then the default allocation $\mathbf{0}$ is implemented.

In the first round the bargaining procedure starts in state 1. In every subsequent round the state evolves according to a Markov process, with transition matrix Q. Let q_{jk} be the probability that given that negotiations are in state j in round t, the state in round $t+1$ is k.

We first introduce some additional notation. When τ rounds remain and negotiations are in state j we define $r_i^{\tau,j}$ for each player i as follows:

- When no rounds remain: $r_i^{0,j} = \mathbf{0}$ for all players i and all states j
- When τ rounds remain: $r_i^{\tau,j} = q_{ik} m_i(r^{\tau-1,i}, S) + \sum_{j \neq i} q_{jk} r_j^{\tau-1,k}$ for all players i and all states j

When it is clear from the context which bargaining problem (S, d) is being referred to, we abuse notation by writing $\hat{m}^{\tau,i} = m_i(r^{\tau,i}, S)$. Moreover we write (a_i, b_{-i}) to refer to a vector with the i'th element equal to a_i and the j'th element equal to b_j whenever $i \neq j$. We first prove a preliminary lemma:

Lemma 3.3.1.
The utility allocation $r^{\tau,j}$ is in the feasible set: $r^{\tau,j} \in S$ for all j and for all τ

Proof. The proof follows by induction. The base case is trivial, since $r^{0,j} = \mathbf{0} \in S$. For the inductive step assume $r^{\tau,j} \in S$. Define $s^{\tau,i} = (\hat{m}^{\tau,i}, r^{\tau,i}_{-i})$ and note that $s^{\tau,i} \in S$. Using the inductive assumption we get:

$$\begin{aligned} r_i^{\tau+1,j} &= \sum_{k \neq i} q_{jk} r_i^{\tau,k} + q_{ji} \hat{m}^{\tau,i} \\ &= \sum_{k \neq i} q_{jk} s^{\tau,k} + q_{ji} s^{\tau,i} \end{aligned}$$

Hence $r^{\tau+1,j} = \sum_{k \in N} q_{j,k} s^{\tau,k}$. Since all vectors $s^{\tau,k} \in S$ and S is a convex set, it follows that $r^{\tau+1,j} \in S$ □

Using this lemma, we now show that the following is a subgame perfect equilibrium (SPE):

Proposition 3.3.2 (SPE with immediate acceptance)**.**
The bargaining model with transition matrix Q has the following SPE. If τ additional rounds remain before the default is implemented and negotiations are in state $i \leq n$, then player i proposes $(\hat{m}^{\tau,i}, r^{\tau,i}_{-i})$. This offer is accepted by all other players.

The proof follows by induction and can be found in the appendix, where we show that players have no profitable deviations in any subgame.

Here $r_i^{\tau,j}$ is player i's expected utility of continuing negotiations, and the minimum utility level he is prepared to accept. The proposer j offers each player i this reservation utility $r_i^{\tau,j}$, while assigning himself the remainder $m_j(r^{\tau,j}, S)$. Moreover the proposer at least weakly prefers this allocation to continuing to the next round, while the other players are indifferent.

If player 1 is strictly risk averse and receives a strictly positive utility allocation $\hat{m}_1^{\tau,1} > 0$, then this is the unique SPE. This is because risk averse proposers strictly prefer to make an acceptable offer rather than continuing to the next round.[5]. Intuitively this result follows from the fact that delaying agreement until close to the deadline increases the risk a player faces and hence a risk averse proposer will strictly prefer to make an immediate agreement. If player 1 is risk neutral or $\hat{m}_1^{\tau,1} = 0$, then there may be other SPEs where player 1 makes an unacceptable initial offer. However these (unusual) SPEs lead to the same expected utility as the one defined above. Therefore we restrict attention to the equilibrium characterised above. We now consider a specific family of transition matrices $Q_0(\lambda, p)$, where $q_{i,j} = p\lambda_j$ whenever $1 \leq j \leq n$ and $q_{i,n+1} = 1 - p$. The two player case is given by:

$$Q_0(\lambda, p) = \begin{pmatrix} p\lambda_1 & p\lambda_2 & 1-p \\ p\lambda_1 & p\lambda_2 & 1-p \\ p\lambda_1 & p\lambda_2 & 1-p \end{pmatrix} \tag{3.1}$$

This transition matrix models a situation where offers are made intermittently. In every round each player has a fixed chance of being selected to be the proposer. Using this simple transition matrix where all the rows are the same, leads to the following result. All remaining proofs an be found in the appendix.

Proposition 3.3.3 (Intermittent Offers)**.**
Consider the bargaining model with transition matrix $Q_0(\lambda, p)$, where proposals are made intermittently. Assume players follow the SPE strategies outlined above. As the number of rounds $T \to \infty$, the utility players obtain converge to their utility level under the (λ, p)-weighted Raiffa solution.

We now consider bargaining models where proposals are made regularly in every round. However instead of each player having a fixed chance of being the proposer in each round, we assume that the player who proposed in round t is more likely to propose again in round $t + 1$. This captures situations where one party is able to revise their offer an uncertain number of times before another party can make a proposal. To model these situations with persistent offers, we define the transition

[5] Meanwhile no SPE exists where player j rejects a proposal when indifferent. This is because in this case the proposer would maximize his utility by proposing the lowest $s_j > r_j^{\tau,1}$

matrix
$Q_1(\lambda, p) := pQ_0(\lambda, 1) + (1-p)I$. In the two player case:

$$Q_1(\lambda, p) = \begin{pmatrix} p\lambda_1 + 1 - p & p\lambda_2 & 0 \\ p\lambda_1 & p\lambda_2 + 1 - p & 0 \\ p\lambda_1 & p\lambda_2 & 1-p \end{pmatrix} \tag{3.2}$$

This model with persistent offers leads to the following result:

Proposition 3.3.4 (Persistent offers)**.**
Consider the bargaining model with transition matrix $Q_1(\lambda, p)$, where the identity of the proposer is persistent. Assume players follow the SPE strategies outlined above. As the number of rounds $T \to \infty$, the expected utility players obtain converge to their utility level under the (λ, p)-weighted Raiffa solution.

Hence the family of weighted Raiffa solutions can be implemented either when offers are made intermittently, or when the identity of the proposer is persistent. The theorem below shows that these solutions can also be implemented by models combining both these features:

Theorem 3.3.5.
Consider any model with transition matrix $Q_\mu(\lambda, p) = \mu Q_0(\lambda, p) + (1-\mu)Q_1(\lambda, p)$ where $\mu \in [0, 1]$. Assume players follow the SPE strategies outlined above. As the number of rounds $T \to \infty$, the utility players obtain converge to their utility level under the (λ, p)-weighted Raiffa solution.

Proof. Define $d(0) = 0$ and $d_i(\tau) := (1 - \lambda_i p) + \lambda_i p m_i(S, d(\tau - 1))$. Note that $d(\tau)$ converges to the (λ, p)-Raiffa solution, as $\tau \to \infty$. After appealing to the immediate acceptance lemma, it remains to be shown that for all τ:

$$d_i(\tau) = r_i^{\tau,j} \;\; \forall j \neq i$$

We prove this by induction. The base case is trivial. From their respective definitions, $r^{0,j} = d(0) = 0$ for all j.
Now consider the inductive step. Suppose $d_i(\tau) = r_i^{\tau,j} \;\forall\; j \neq i$.

$$
\begin{aligned}
r_i^{\tau+1,j} &= \mu\Big(\sum_{k=1,k\neq i}^{n} \lambda_k p r_i^{\tau,k} + \lambda_i p \hat{m}_i^{\tau,i} + (1-p) r_i^{\tau,n+1}\Big) \\
&\quad +(1-\mu)\Big(\sum_{k=1,k\neq i}^{n} \lambda_k p r_i^{\tau,k} + \lambda_i p \hat{m}_i^{\tau,i} + (1-p) r_i^{\tau,j}\Big) \\
&= \sum_{k=1,k\neq i}^{n} \lambda_k p r_i^{\tau,k} + (1-p) r_i^{\tau,n+1} + (1-\mu)(1-p)) r_i^{\tau,j} + \lambda_i p m_i(S, r^{\tau,i}) \\
&= (1-\lambda_i p) d_i(\tau) + \lambda_i p m_i(S, d(\tau)) \\
&= d_i(\tau+1)
\end{aligned}
$$

The first line states the definition of $r_i^{\tau+1,j}$. Rearranging and collecting terms leads to the second line. The third line uses the induction hypothesis, using the fact that reservation utilities are the same in every state. In the final step, the definition of d is applied. Since $m_i(S, d(\tau) \geq d_i(\tau)$ it follows that when τ rounds remain, all players receive at least $d_i(\tau)$. Hence as the number of rounds remaining $\tau \to \infty$, the utility players obtain converge to $d(\tau)$. Therefore if T is large, then the (λ, p)-Raiffa solution is approximately implemented. □

These implementations show that the (λ, p)-Raiffa solution arises from a number of simple bargaining models, where offers are made intermittently or the identity of the proposer is persistent. In particular any (λ, p)-Raiffa solution can be implemented by a simple bargaining model without discounting, where in every round players have a fixed chance of being the proposer regardless of past history. This provides a strong non-cooperative foundation for this family of weighted bargaining solutions.

3.4 Discussion

When players are risk neutral, the bargaining weights correspond to the proportion of cake each player is allocated in equilibrium as is the case under the Nash and the KS solution. Hence the solution concepts coincide when players are risk neutral.

We have shown that the (λ, p)-Raiffa solution can be implemented in simple games using discrete time. When players have equal bargaining weights and $p \to 0$ which means that offers are made rarely, the model presented can approximately implement the continuous Raiffa solution. In the two player case, this is done using the following Q-matrix, where $\epsilon \to 0$.

$$Q^{\epsilon}(\lambda, p) = \begin{pmatrix} \epsilon & \epsilon & 1-2\epsilon \\ \epsilon & \epsilon & 1-2\epsilon \\ \epsilon & \epsilon & 1-2\epsilon \end{pmatrix}$$

This discrete time model is related to the continuous time model studied by Ambrus and Lu (2015a), who consider offers arriving according to a Poisson process. This suggests that the λ-continuous Raiffa solution and related weighted solutions may also arise in continuous time settings.

Finally the implementations introduced have similarities with implementations of the weighted Nash solution. In particular Britz et al. (2010) show that a weighted Nash solution arises in settings similar to those considered above, where there are infinitely many rounds and players discount at a rate tending to one. We conjecture that if a discount factor is added to the finite horizon game described above, the weighted Nash solution will be implemented whenever $\beta \to 1$, $\beta^T \to 0$ and Q is irreducible. This result would bridge the infinite horizon result proved in Britz et al. (2010) and the finite horizon result proved here.

Currently the weighted Nash solution is the standard solution used in applications by economists. However we argue here that the weighted (λ, p)-Raiffa solutions also have strong foundations and may in some situations be preferred, particularly in settings where discounting is unimportant.

3.5 Appendix

3.5.1 Literature Summary:

Table 3.2. Literature summary

		Nash		Kalai-Smorodinsky		Raiffa $p = \{0, 1\}$		Raiffa $p = (0, 1)$	
SYMMETRIC	Axiomatic	Nash	(1953)	Kalai-Smorodinsky	(1970)	Raiffa	(1953)	Diskin et al	(2011)
VERSION	Simultaneous	Nash	(1953)	Moulin	(1984)	-	-	-	-
	Sequential	Binmore et al	(1970)	-	-	Myerson Sjostrom $p = 1$ Gomes	(1991) (1991) (1998)	Diskin et al	(2011)
ASYMMETRIC	Axiomatic	Chung & Ely	(1975)	Dubra This paper	(2001)	This paper		This paper	
VERSION	Simultaneous	Carlsson	(1991)	-	-	-	-	-	-
	Sequential	Britz et al	(2011)	-	-	This paper		This paper	

3.5.2 Additional notation

We first introduce some additional notation. Throughout the appendix we use bold numbers to refer to a vector with n elements all of which are equal to the bold number. Hence $\mathbf{1}$ represents the unit vector. We use $\mathbf{k} \in \mathrm{Re}^n$ to refer to a generic vector such that $\mathbf{k}_i = k$ for all i. Moreover a bold letter $\mathbf{v}$ is used to refer to any generic vector $\mathbf{v} = (v_1, \ldots, v_n) \in \mathrm{Re}^n_{++}$.

3.5.3 Consequences of scale invariance

Define a such that $a\mathbf{1} = \mathbf{a} \in PF(S)$ and define $\hat{\Delta}(\lambda) = CH\{\mathbf{0}, (\frac{1}{\lambda_i}, \mathbf{0}_{-i})\}$. Using these definitions let $T_{a,\lambda} = CH(\hat{\Delta}(\lambda), \mathbf{a})$. We first prove a lemma related to these convex sets. It says that if the process starts on a path towards $\mathbf{a}$ then it does not change direction.

Lemma 3.5.1. *Suppose Φ is associated with a partial solution δ which satisfies δ-monotonicity. Moreover suppose $\delta(T_{a,\lambda}, \mathbf{0}) = p\mathbf{a}$. Then $\Phi(T_{a,\lambda}, \mathbf{0}) = \boldsymbol{a}$.*

Proof. Consider the following linear mapping M: $\lambda \mapsto \lambda$ and $\delta(S, \mathbf{0}) \mapsto \mathbf{0}$. Now note that $M(S_{\delta(S,\mathbf{0})}) = S$. Hence by scale invariance:

$$\begin{aligned} M\Big(\Phi(S_{\delta S,d}, \delta(S, \mathbf{0}))\Big) &= \Phi(S, \mathbf{0}) \\ M\Big(\Phi(S, \mathbf{0})\Big) &= \Phi(S, \mathbf{0}) \end{aligned}$$

But since the linear mapping M has a unique fixed point, given by λ, it follows that: $\Phi(S, \mathbf{0}) = \lambda$.

□

Lemma 3.5.2. *If $\Phi(\Delta(\lambda), \mathbf{0}) = \mathbf{1}$, then $\delta(\Delta(\lambda), \mathbf{0}) = p\mathbf{1}$*

Proof. Let $\delta(\Delta(\lambda), \mathbf{0}) = p\mathbf{a}$ where $\mathbf{a} \in PF(\Delta(\lambda))$. By the lemma above $\Phi(\Delta(\lambda), \mathbf{0}) = \mathbf{a}$. Hence $\mathbf{a} = 1$ and $\delta(\Delta(\lambda), \mathbf{0}) = p\mathbf{1}$. □

3.5.4 Proof: Weighted Raiffa solution

We split the proof of proposition 3.2.2 into two parts for clarity. The second part is similar to the proof of the symmetric case which can be found in Diskin et al. (2011). However since the axioms considered here are weaker - in particular the Φ-invariance axiom is weaker than the δ-invariance axiom used by Diskin et al. (2011), a longer proof is required.

First the simplex is defined as follows:

$$\Delta(\mathbf{1}) = \{\mathbf{s} \in \mathrm{Re}^n \,|\, \sum_{i=1}^{n} s_i \leq 1 \text{ and } \mathbf{s} \geq 0\}$$

For any vector $\mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathrm{Re}^n_{++}$ the invertible linear transformation $f^{\mathbf{v}} : \mathrm{Re}^n \mapsto \mathrm{Re}^n$ is defined as follows: for any $\mathbf{s} \in \mathrm{Re}^n$, $f^{\mathbf{v}}_i(\mathbf{s}) = v_i s_i$. Using the linear transformation $f^{\mathbf{v}}$ we can now define the stretched simplex $\Delta(\mathbf{v})$ as follows:

$$\Delta(\mathbf{v}) = f^{\mathbf{v}}(\Delta(\mathbf{1}))$$

In the first section of the proof, we show that for some $p \in [0,1]$ and some $\lambda \in \mathrm{Re}^n_+$, $\frac{1}{k}\delta(\Delta(\mathbf{k}), 0) \to p\lambda$ as $k \to \infty$. We can then use this fact in the second part of the proof and avoid using the stronger axiom of δ-invariance.

Lemma 3.5.3. $\frac{1}{k}\delta(\Delta(\boldsymbol{k}), \boldsymbol{0}) \to p\lambda$ *as* $k \to \infty$ *for some* $\lambda \in [0,1]^n$ *and* $p \in (0,1]$.

Proof. Note that $\delta(\Delta(\mathbf{v}), \mathbf{0}) = g(\mathbf{v}) f^{\mathbf{v}}(\lambda)$ where $g(\mathbf{v}) \in \mathrm{Re}$ is a constant multiplying each element in $f^{\mathbf{v}}(\lambda)$. If not it follows from Φ-invariance that $\Phi(\Delta(\mathbf{v}), \mathbf{0}) \neq \Phi(\Delta(\mathbf{v}), \delta(\Delta(\mathbf{v}), \mathbf{0}))$ and this violates the fact that δ is associated with Φ.

Note that $f^{\mathbf{k}}(\lambda) = k\lambda$ and hence $\frac{1}{k}\delta(\Delta(\mathbf{k}), \mathbf{0}) = g(\mathbf{k})\lambda$.

Take $k' > k$ and define $\mathbf{v} = (k'_1, k_{-1})$. Now $\Delta(\mathbf{k}) \subset \Delta(\mathbf{v}) \subset \Delta(\mathbf{k'})$, and so by δ-monotonicity $\delta(\Delta(\mathbf{k}), \mathbf{0}) \leq \delta(\mathbf{v}, 0) \leq \delta(\Delta(\mathbf{k'}), \mathbf{0})$.

Using the definition above, it follows that $g(\mathbf{k})f^{\mathbf{k}}(\lambda) \leq g(\mathbf{v})f^{\mathbf{v}}(\lambda) \leq g(\mathbf{k}')f^{\mathbf{k}'}(\lambda)$. Now the first inequality implies that $g(\mathbf{k}) \leq g(\mathbf{v})$. Suppose this was not true. Then, since $f^{\mathbf{v}}_n = f^{\mathbf{k}}_n$ the last component of $\delta(\Delta(\mathbf{k}), \mathbf{0})$ is higher than the last component of $\delta(\Delta(\mathbf{v}), \mathbf{0})$. This violates δ-monotonicity. Meanwhile the second inequality implies that $g(\mathbf{v}) \leq g(\mathbf{k}')$. Suppose this was not true. Then since $f^{\mathbf{v}}_1 = f^{\mathbf{k}'}_1$, the first component of $\delta(\Delta(\mathbf{v}), \mathbf{0})$ is higher than the first component of $\delta(\Delta(\mathbf{k'}), \mathbf{0})$. This violates δ-monotonicity.

Putting these inequalities together implies $g(\mathbf{k}) \leq g(\mathbf{v}) \leq g(\mathbf{k}')$. Note that $g(\mathbf{k})$ is increasing in each of its arguments. Moreover $g(\mathbf{k})$ is bounded above by 1, since otherwise $\delta(\Delta(\mathbf{k}), 0) \notin \Delta(\mathbf{k})$. Furthermore $g(\mathbf{k})$ must be positive, or $\delta(\Delta(\mathbf{k}), \mathbf{0}) \notin \Delta(\mathbf{k})$, which violates the requirement that δ is feasible. Finally $g(\mathbf{k}) \neq 0$, because otherwise repeated applications of the interim agreement function would not change the interim agreement point and hence the pareto optimal solution would never be reached. Hence $g(\mathbf{k}) \in (0, 1]$ and is increasing in each of its elements k_i for ß $\in \{1, \ldots, n\}$.

Therefore as $k \to \infty$ and considering the particular sequence of $\mathbf{k}$, it must be the case that for some $p \in (0, 1]$, $g(\mathbf{k}) \to p$. Since $\delta(\Delta(\mathbf{k}), \mathbf{0}) = g(\mathbf{k})f^{\mathbf{k}}(\lambda)$ and $f^{\mathbf{k}}(\lambda) = k\lambda$, it follows that $\frac{1}{k}\delta(\Delta(\mathbf{k}), \mathbf{0}) \to p\lambda$.

□

Lemma 3.5.4. *If* $\mathbf{0} \in S$ *and* $m(S, \mathbf{0}) = \mathbf{1}$, *then* $\Phi(S, \mathbf{0}) = \Phi(S, p\lambda)$

Proof. Take any normalised bargaining problem $(S, \mathbf{0})$ such that the ideal point $m(\mathbf{0}) = \mathbf{1}$. In order to show the result, it is sufficient to prove that for any degree of accuracy $\bar{\epsilon} > 0$ there exists an interim agreement d^1 such that $||d^1 - p\lambda|| < \bar{\epsilon}$ and $\Phi(S, \mathbf{0}) = \Phi(S, d^1)$.

First - using the earlier result - pick K such that for all $k > K$, $(p - \epsilon)\lambda \leq \frac{1}{k}\delta(\Delta(\mathbf{k}), \mathbf{0}) \leq p\lambda$. Define $S(k) = f^{\mathbf{k}}(S)$ and consider the stretched simplex $\Delta(\mathbf{k}) \subseteq S(k)$. By δ-monotonicity, $\delta(\Delta(\mathbf{k}), \mathbf{0}) \leq \delta(S(k), \mathbf{0})$. Since $\frac{1}{k}\delta(\Delta(\mathbf{k}), \mathbf{0}) \geq (p - \epsilon)\lambda$, it follows that $\delta(S(k), \mathbf{0}) \geq k(p - \epsilon)\lambda$.

Secondly for $a \in \mathrm{Re}$ where $a > 2$ define $\mathbf{v}(a,i)$ as follows. $\mathbf{v}_i(a,i) = \frac{ak}{a-1}$ and whenever $j \neq i$ $\mathbf{v}_j(k,i) = ak$. Note that for some scalar $b \in \mathrm{Re}_+$, $\delta(\Delta(\mathbf{v}(a,i)), \mathbf{0}) = bf^{\mathbf{v}(a,i)}(\lambda)$. If not the bargaining solution would violate Φ-invariance, or the interim agreement function δ would not be associated with Φ.

Now note that $\Delta(\mathbf{v}(a,i)) \subseteq \Delta(\mathbf{ak})$. Using δ-monotonicity for all $j \neq i$:

$$\begin{aligned}
\delta_j(\Delta(\mathbf{v}(a,i)), \mathbf{0}) &= bf_j^{\mathbf{v}(a,i)}(\lambda) \\
&= bak\lambda_j \\
&\leq \delta_j(\Delta(\mathbf{ak}), \mathbf{0}) \\
&\leq pak\lambda_j
\end{aligned}$$

It follows that $b \leq p$. But also note that $S(k) \subseteq \Delta(\mathbf{v}(a,i))$, and so by δ-monotonicity:

$$\begin{aligned}
\delta_i(\Delta(\mathbf{v}(a,i)), \mathbf{0}) &= bf_i^{\mathbf{v}(a,i)}(\lambda) \\
&= \frac{bak}{a-1}\lambda_i \\
&\geq \delta_i(S(k), \mathbf{0})
\end{aligned}$$

Since a can be arbitrarily high, it follows that $\delta(S(k), \mathbf{0}) \leq bk\lambda \leq pk\lambda$.

Finally define $d^1 = \frac{1}{k}\delta(S(k), \mathbf{0})$. Using Φ-invariance twice note that:

$$
\begin{aligned}
f^{\mathbf{k}}\Big(\Phi(S,\mathbf{0})\Big) &= \Phi(f^{\mathbf{k}}(S), f^{\mathbf{k}}(\mathbf{0})) \\
&= \Phi(S(k), \mathbf{0}) \\
&= \Phi(S(k), \delta(S(k), \mathbf{0})) \\
&= \Phi(S(k), kd^1) \\
&= \Phi(f^{\mathbf{k}}(S), f^{\mathbf{k}}(d^1)) \\
&= f^{\mathbf{k}}\Big(\Phi(S, d^1)\Big)
\end{aligned}
$$

Since $f^{\mathbf{k}}$ is invertible, this implies that $\Phi(S,\mathbf{0}) = \Phi(S,d^1)$. Also note from the two results above that $k(p-\epsilon)\lambda \leq \delta(S(k),\mathbf{0}) \leq kp\lambda$, and hence $(p-\epsilon)\lambda \leq d^1 \leq p\lambda$. Since ϵ is arbitrarily small, this proves the result. □

3.5.5 Proof: Weighted Kalai-Smorodinsky solution

Proof: 3.2.4 Weighted Kalai-Smorodinsky solution

Proof. Take a normalised bargaining problem $(S,\mathbf{d})$ such that:

1. $\mathbf{d} = \mathbf{0}$
2. $m_i(S,\mathbf{0}) = \frac{1}{\lambda_i}$

Define a such that $a\mathbf{1} = \mathbf{a} \in PF(S)$. Moreover define $\hat{\Delta}(\lambda) = CH\{\mathbf{0}, (\frac{1}{\lambda_i}, \mathbf{0}_{-i})\}$. Note that $\Phi(\Delta,\mathbf{0}) = \lambda$ and hence by - scale invariance - $\Phi(\hat{\Delta}(\lambda),\mathbf{0}) = \mathbf{1}$.

Finally define $T := CH\{S,\mathbf{a}\}$. Appealing to the two lemmas, for some scalars $p > 0$ and $p' > 0$:

$$
\begin{array}{rcll}
\Phi(\hat{\Delta}(\lambda), \mathbf{0}) & = & \mathbf{1} & \text{(scale invariance argument above)} \\
\delta(\hat{\Delta}(\lambda), \mathbf{0}) & = & \text{p } \mathbf{1} & \text{(using lemma 3.5.2)} \\
\delta(T, \mathbf{0}) & = & \text{p' } \mathbf{1} & \text{(δ-monotonicity lemma)} \\
\Phi(T, \mathbf{0}) & = & \mathbf{a} & \text{(using lemma 3.5.1)} \\
\Phi(S, \mathbf{0}) & = & \mathbf{a} & \text{(Φ-monotonicity)} \\
\Phi(S, \mathbf{0}) & = & \Phi_\lambda^{KS}(S, \mathbf{0}) & \text{(definition of Φ_λ^{KS})}
\end{array}
$$

This proves that $\Phi(S, \mathbf{0}) = \Phi_\lambda^{KS}(S, \mathbf{0})$, whenever S is suitably normalised. The full result follows by appealing to scale invariance.

□

3.5.6 Proofs: Non cooperative foundation

Proof: 3.3.2 SPE

Proof. The proof follows by induction. We show that players have no profitable deviations in any subgame. Consider the base case. WLOG suppose negotiations are in state i in the last round.

1. Any player $j \neq i$ is indifferent between accepting (receiving $s_j = 0$) and rejecting (receiving $s_j = 0$). Hence rejecting is not a profitable deviation.

2. Suppose player i makes a lower offer, where for one player $s_j < 0$. In this case player j rejects and player i receives $d_i = 0 \leq m_i(S, \mathbf{0})$. Hence making a lower offer is not a profitable deviation.

3. Suppose player i makes a higher offer where $s_j > 0$ and $s_{-j} \geq 0$. The offer is accepted, but since, $m_i()$ is strictly decreasing in the utilities of other players $m_i(S, (\mathbf{0}_{-j}, x_j)) < m_i(S, \mathbf{0})$. Hence making a higher offer is not a profitable deviation.

Hence if negotiations are in state $i \leq n$ in the last round, then player i proposes $(\hat{m}^{0,i}, r_{-i}^{0,i})$ and this offer is accepted

Now, consider the inductive step. Suppose that when τ rounds remain player i proposes $(\hat{m}^{\tau,i}, r_{-i}^{\tau,i})$.

1. If player $j \neq i$ rejects, the process will move to the next round and will enter state k with probability $q_{i,j}$. By the inductive step, in the next round the offer $(\hat{m}^{\tau,k}, r^{\tau,k}_{-k})$ will be accepted. Hence if player j rejects he expects to receive $\sum_{k \neq j} q_{ik} r_i^{k,\tau} + q_{ij} m_i(r^{\tau,j}) = r_i^{\tau,j}$, So player $j \neq i$ is indifferent between accepting and rejecting. Hence rejecting is not a profitable deviation.

2. If player i makes any player a lower offer, where $s_j < r_j^{\tau,i}$ player j rejects. In this case - by using the same argument as above, player i expects to receive $r^{\tau,i}$ in the next round. Since $r^{\tau,i} \in S$, it follows that $\hat{m}^{\tau,i} \geq r^{\tau,i}$. Hence making a lower offer is not a profitable deviation.

3. If player i makes a higher offer $\mathbf{s}$ where $s_j > r_j^{\tau,i}$ and $s_{-i} \geq r^{\tau,i}_{-i}$, then the offer is accepted. However since $m_i()$ is strictly decreasing in the utilities of other players $m_i(S, \mathbf{s}) < m_i(S, r^{\tau,i})$. It follows that $s_i < \hat{m}^{\tau,i}$, and so making a higher offer is not a profitable deviation.

Hence if negotiations are in state $i \leq n$ with τ rounds remaining, then player i proposes $(\hat{m}^{\tau,i}, r^{\tau,i}_{-i})$ and this offer is accepted. $\square$

Chapter 4

A Non-Cooperative Foundation for the Continuous Raiffa Solution

4.1 Introduction

Almost simultaneously to the seminal work of Nash (1950), Nash (1953), Raiffa (1953) formulated two closely related alternatives to the Nash bargaining solution. The first – the *discrete* or *sequential* Raiffa solution – is defined as the limit of a series of intermediate agreements that is constructed by an iterative random dictator procedure. The second solution he proposed – the *continuous* Raiffa solution – is in the same spirit, but assumes that the step size between intermediate agreements is infinitesimally small. As such, it is obtained as the endpoint of a continuous intermediate agreement *curve*. The advantage of this second approach is that the resulting solution remains well-defined if bargaining problems are allowed to be non-convex. Indeed, while it is standard to assume that the feasible sets in the bargaining problem are convex, there is a wide literature that recognizes that this requirement may in many instances be too strict.[1] As such, we do not require that bargaining problems have convex feasible sets.

A prolific strand of literature in economics, known as the *Nash program*, focuses on non-cooperative foundations for cooperative solution concepts. More specif-

[1] See among others Conley (1991), Conley and Wilkie (1994), Conley and Wilkie (1996), Zhou (1997), Xu (2006), Xu and Yoshihara (2013)

ically, this agenda aims at constructing reasonable non-cooperative games such that their (unique) equilibrium payoffs yield, or somehow approximate, the outcome of the considered cooperative solution concept. While several results provide non-cooperative support for the discrete Raiffa solution, the continuous version has received much less attention.[2] With this paper we aim to address this gap in the literature. In particular, we provide a direct support result for the continuous Raiffa solution.

The game constructed to this end is an n-player bargaining model in the tradition of Stahl (1972) and Rubinstein (1982). Players are assumed to make proposals that are instantaneously accepted or rejected by all opponents; in case of unanimous agreement, the proposal is implemented, otherwise it is rejected and the game continues. Apart from these common features, the game differs from the classic models in several important ways: bargaining occurs in continuous time and the game features a finite deadline that ends the negotiations with all players obtaining zero payoffs. Moreover, the timing of the proposals is stochastic in the sense that they are governed by independent player-specific Poisson processes. It turns out that this game has a unique subgame perfect equilibrium (SPE) in which players use Markovian strategies, and the first proposal made is accepted. The main result of this paper is that the payoffs players realize in this SPE converge to the continuous Raiffa solution as the horizon tends to infinity.

Ambrus and Lu (2015b) consider a *coalitional* version of the above-described game, in the spirit of Okada (1996), Okada (2011), Chatterjee and K. (1993) and Yan (2003). The key distinction between the two versions lies in their respective underlying cooperative problems: Ambrus and Lu (2015b) take this to be a convex TU game, whereas the present paper assumes it is a pure bargaining problem. As is well-known, both are special cases of the more general class of NTU games (see a.o. Hart (2004)). Furthermore, Ambrus and Lu (2015b)'s framework assumes that players are impatient in the sense that utilities are discounted over time. These distinctions are important. Where we obtain non-cooperative support for the continuous Raiffa solution, as described above, Ambrus and Lu (2015b) obtain a non-cooperative foundation for the core of the underlying TU game.

[2] A notable exception is Diskin et al. (2011), who provide support for solutions that approximate the continuous Raiffa solution. See the discussion of the related literature below.

Finally, it is demonstrated that the support result does not depend in full on the chosen proposer protocol. In particular, we adopt a variation of the well-known *rejector-proposes* protocol, studied by Selten (1981), Chatterjee and K. (1993) and Britz and Predtetchinski (2012) among others, and show that this does not affect the SPE, nor the associated payoffs, nor its limit as the horizon tends to infinity.

Related Literature The Nash program literature on the Raiffa solution has primarily focused on its discrete version. Myerson (1991)(pp. 393-394) describes a two-player, discrete- and finite-time, random-recognition bargaining game that can be regarded a discrete-time analogue of the game considered in this paper. The payoffs associated with the unique SPE of this game converge to the *discrete* Raiffa solution, as the number of bargaining rounds T diverges to infinity.

Sjostrom (1991) proposes a similar game with the assumption that payoffs are discounted with factor r. This game too has a finite deadline that ends negotiations, and actions take place at T equidistant time points within this fixed time interval. Sjostrom (1991) demonstrates that the unique SPE payoffs of this game converge to an outcome within a certain distance from the discrete Raiffa solution, as the partition of the bargaining interval $[0, T]$ becomes more and more refined.

Diskin et al. (2011) introduce a class of *generalized* Raiffa solutions for n players; each such solution corresponds to the limit point of a series of intermediate agreements, where the step size between agreements lies within the interval $(0, 1/n]$. They provide a non-cooperative foundation for their solution class that is again based on Myerson's game. Of course, this support result only holds for generalized solutions with strictly positive step size, and thus necessarily entails an approximation that is absent from the model in this paper. To our knowledge this paper provides a first *direct* support result for the continuous Raiffa solution.

Structure of the Paper The remainder of the paper is organized as follows. Section 2 introduces the bargaining problem and the continuous Raiffa solution. Section 3 describes and analyzes the non-cooperative bargaining game. In Section 4 it is demonstrated that the payoffs associated with the unique SPE of this game converge to the continuous Raiffa solution as the horizon tends to infinity. Section 5 considers an alternative proposer protocol, and Section 6 concludes.

4.2 Preliminaries

4.2.1 The Bargaining Problem

A *bargaining problem* is defined by a finite set of players $N := \{1, \ldots, n\}$ with $n \geq 2$, and a subset S of $\mathbb{R}^n$, that is closed and strictly comprehensive (i.e. $y \in S$ and $x \leq y$ implies $x \in S$; if $x \neq y$, then $z > x$ for some $z \in S$)[3], that contains an outcome $z > \mathbf{0} =: (0, \ldots, 0)$, and is such that $S \cap \mathbb{R}^n_+$ is bounded. It is further assumed that S satisfies the following condition:[4]

(A1): There exists a $K > 0$ such that for all $i, j \in N$, and for all $x, y \in \partial S \cap \mathbb{R}^n_+$ with $x_{-\{i,j\}} = y_{-\{i,j\}}$: $|x_i - y_i| \leq K|x_j - y_j|$.

Note that we do not insist on convexity of S.

Fixing the set of players N, a bargaining problem is henceforth denoted by its feasible set S. The class of all bargaining problems S is denoted $\mathcal{B}$. A *bargaining solution* is a map $\varphi : \mathcal{B} \to \mathbb{R}^n$ that assigns to each bargaining problem $S \in \mathcal{B}$ a unique outcome $\varphi(S) \in S$.

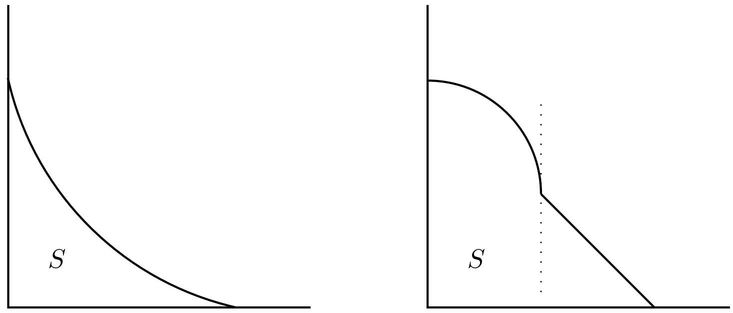
The interpretation of the bargaining problem is as follows. An outcome $x \in \mathbb{R}^n$ represents a utility allocation, in the sense that each x_i is the utility payoff obtained by player i; the *feasible set* S represents the set of all utility allocations players in N can jointly realize; players must find agreement on an outcome $x \in S$, and failure to do so leads to the unfavorable outcome $\mathbf{0}$. The point $\mathbf{0}$ is therefore also called the *disagreement point.*[5] The solution outcome $\varphi(S)$ is interpreted as the compromise reached by the players in N when faced with the problem S. Condition (A1) says that if an agent i gives up some of his utility $\varepsilon > 0$, then there is an upper bound $K\varepsilon$ on the associated compensation other agents (i.e., $j \in N \setminus \{i\}$) can feasibly attain.

Condition (A1) is a rather mild assumption. For instance, if the bargaining problem is a utility representation of simple economic division problem, then the condition already holds if the agents' utility functions are continuously differen-

[3] For $x, y \in \mathbb{R}^n$, $x \geq y$ is taken to mean $x_i \geq y_i$ for all $i \in N$; the vector inequalities $>$, $\leq$ and $<$ are similarly defined.

[4] For a closed set $S \in \mathbb{R}^n$, $\partial S := S \setminus \mathrm{int}(S)$, where $\mathrm{int}(S)$ denotes the interior of S.

[5] Normalization of the disagreement point to the zero vector $\mathbf{0}$ is without loss of generality.



(a) S satisfies (A1) (b) S violates (A1)

Figure 4.1. An illustration of condition (A1).

tiable. It is worth pointing out that Condition (A1) is not implied by convexity. Examples demonstrating this are easily constructed.

4.2.2 A Family of Raiffa Solutions

In order to define our solutions of interest, we first formalize the notion of a *maximal claims vector*. Given a bargaining problem $S \in \mathcal{B}$ and an outcome $x \in S \cap \mathbb{R}^n_+$, let $m(x, S) := (m_1(x, S), \ldots, m_n(x, S))$, where

$$m_i(x, S) := \max\{y_i | (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) \in S\}$$

for all $i \in N$. Each $m_i(x, S)$ is the maximal claim player i holds over the surplus that remains of S, given that an intermediate agreement has been reached on the outcome x. Note that $m(\cdot, S)$ is a well-defined vector function by strict comprehensiveness of S and compactness of the set $S \cap \mathbb{R}^n_+$. Whenever the problem S is understood, we write $m(x)$ rather than $m(x, S)$.

Given a convex problem $S \in \mathcal{B}$, the *discrete* Raiffa solution (Raiffa (1953)) is then defined as the limit of the sequence $\{x^t\}_{t=0}^{\infty}$, where $x^0 = \mathbf{0}$ and

$$x^{t+1} := x^t + \frac{1}{n}(m(x^t) - x^t) \tag{4.1}$$

for all $t \geq 1$. It is based on the intuitive notion that agreement is found on the midpoint of all the maximal claims agents hold over the surplus to divide. If this midpoint is not efficient, then agents again stake out their maximal claim over the surplus that remains, and reach a next compromise on the midpoint of those claims. The solution outcome is reached by iteratively applying this reasoning, until the entire surplus is allocated.

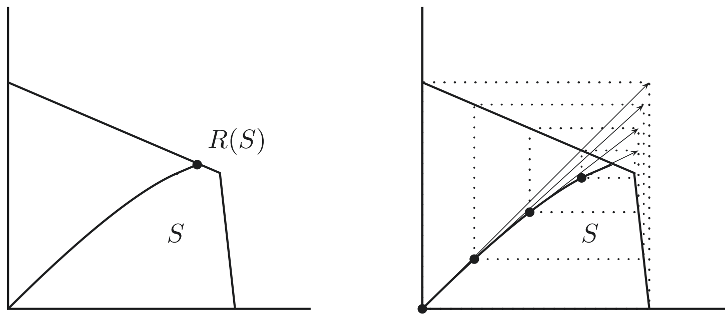


Figure 4.2. An illustration of the continuous Raiffa solution.

Note that if the problem S is not convex, then the midpoint of all maximal claims need not be feasible, and the discrete Raiffa solution may not be well-defined. This problem can be addressed by decreasing the step size $1/n$ in (4.1) to some $0 < c < 1/n$, as proposed by Diskin et al. (2011), or to 0, as proposed by Raiffa (1953). In the latter case, (4.1) becomes an initial value problem, and the sequence of intermediate agreements becomes an intermediate agreement *curve*. The limit point of this curve is our solution of interest, the *continuous Raiffa solution*. It has been considered for convex two-player problems by Raiffa (1953), Livne (1989), Peters and Damme (1991) among others; in these studies, the intermediate agreement curve is obtained as the solution of the initial value problem $dx_1/dx_2 = (m_1(x,S) - x_1)/(m_2(x,S) - x_2)$ with the initial condition $x_1(0) = 0$. In a multiple-player setting, the intermediate agreement curve could similarly be obtained by parameterizing the utilities of players $i \in N \setminus \{n\}$ in terms of the utility of player n. However, in their discussion of the continuous Raiffa solution, Diskin et al. (2011) explicitly used the continuous version of (4.1). Their definition can be generalized to a class of *weighted* Raiffa solutions.

Definition 23. *For* $\mu := (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n_{++}$*, the* continuous Raiffa solution $R^\mu : \mathcal{B} \to \mathbb{R}^n$ *is defined as*

$$R^\mu(S) := \lim_{t\to\infty} x(t)$$

where $x : [0, \infty) \to S$ *is the unique solution of the Initial Value Problem*[6]

$$x'(t) = \mu(m(x(t)) - x(t)) \quad \text{and} \quad x(0) = \mathbf{0}. \tag{4.2}$$

The class $\mathcal{R} := \{R^{\mu} \mid \mu \in \mathbb{R}^n_{++}\}$ *contains all such solutions.*

Up to a constant $c > 0$, $R^{(c,\dots,c)}$ is the unique symmetric solution in $\mathcal{R}$; this solution is also denoted R. An argument similar to Theorem 5 of Diskin et al. (2011) shows that all solutions in $\mathcal{R}$ are well-defined.

Proposition 4.2.1. *For all* $S \in \mathcal{B}$ *and* $\mu \in \mathbb{R}^n_{++}$, *problem* (4.2) *has a unique solution* $x : [0, \infty) \to \mathbb{R}^n$, *and* $\lim_{t\to\infty} x(t)$ *is contained in the boundary of* S.

All proofs are relegated to the Appendix.

4.3 A Non-Cooperative Bargaining Game

We consider a continuous-time bargaining game with stochastic timing of proposals and a finite deadline, similar to the game proposed by Ambrus and Lu (2015b). The underlying framework of this game is a bargaining problem $S \in \mathcal{B}$, as defined in the previous section. Bargaining occurs in a continuous time interval $[0, T]$, where the *deadline* T is finite and known to all players. For each player $i \in N$, the opportunity to make proposals is produced by a Poisson process with player-specific arrival rate $\lambda_i > 0$. These processes are assumed to be independent, and the associated arrival rates are further assumed to sum to one. The latter is without loss of generality, since the interval in which bargaining occurs can always be rescaled.

Whenever player i's process realizes, he proposes an allocation $x \in S$. Opponents then make instantaneous and sequential accept/reject decisions concerning this proposal. It is assumed that the order in which players decide on the made proposal corresponds with their indices – i.e. player k's decision precedes player l's decision if and only if $k < l$.[7] If all of i's opponents accept, then bargaining

[6] For $x, y \in \mathbb{R}^n$, we define $xy := (x_1y_1, \dots, x_ny_n)$.

[7] The exact order in which players decide on made proposals is irrelevant.

ends with the implementation of the proposal. If at least one player rejects, then the game continues until the next arrival occurs and the above procedure is repeated. If no agreement is reached at or before the deadline T, then bargaining ends, and all players realize their disagreement value 0. A particular game of this form is described by $\Gamma = \{S, \lambda, T\}$, where $S \in \mathcal{B}$ is the underlying pure bargaining problem, $\lambda := (\lambda_1, \ldots, \lambda_n)$ the vector of players' arrival rates, and T the deadline ending negotiations.

4.3.1 Strategies

A strategy in a game $\Gamma = \{S, \lambda, T\}$ consists of two elements: which proposals to make when proposing, and which to accept or reject when responding. Which action a player chooses in either situation may depend on the history of play of the game. Consider a player $i \in N$. If he is the *proposer* at $t \in [0, T]$, then the history includes the times $0 \leq t_1 \leq \ldots \leq t_k < t$ of all previous offers (if any), and for each such time t_l, $l = 1, \ldots, k$, it specifies the corresponding proposal, the corresponding proposer, and the corresponding set of rejectors. If he is the *responder* at $t \in (0, T]$, then the history further includes the time-t proposal, the identity of its proposer, and the (possibly empty) set of rejectors so far.

Denote by H_i^p the set of all histories after which player i must make a proposal, and denote by H_i^r the set of all histories after which he must respond to a proposal. His *strategy* is then described by the pair (f_i, g_i), where $f_i : H_i^p \to S$ maps histories of H_i^p into feasible proposals $x \in S$, and $g_i : H_i^r \to \{Y, N\}$ maps histories of H_i^r into an accept/reject decision on the prevalent offer. A *strategy profile* is a tuple $(f, g) \equiv (f_i, g_i)_{i \in N}$.

4.3.2 Subgame Perfect Equilibrium

Heuristically, the construction of an SPE is based on the idea that proposers make offers such that responders are indifferent between accepting and rejecting. In particular, fixing a game $\Gamma = \{S, \lambda, T\}$, it is assumed that each agent $i \in N$ proposing at time $t \in [0, T]$, offers all opponents $j \in N \setminus \{i\}$ their respective reservation values – denoted $r_j(t)$ – and that he claims $p_i(t)$ for himself. Furthermore, it is assumed that such proposals are accepted.

These assumptions allow us to derive an expression for agents' reservation values. Consider again a player i in N. At any time $t \in [0, T]$ the next realization of any of the n concurrent processes occurs at a time s in the interval $[t, T]$. The probability that it is player i's process that then realizes, is given by λ_i; with probability $1 - \lambda_i$ some other process realizes first. Thus, player i's expected utility payoff at time s is $u_i(s) = \lambda_i p_i(s) + (1 - \lambda_i) r_i(s)$. Since the waiting time until the first next offer is exponentially distributed with rate 1, we obtain the following expression for $r_i(t)$:

$$r_i(t) = \int_t^T e^{-(s-t)} [\lambda_i p_i(s) + (1 - \lambda_i) r_i(s)] ds.$$

After offering all agents $j \neq i$ their reservation values $r_j(t)$, a proposer i claims the utility that makes his proposed allocation efficient. That is, $p_i(t) = m_i(r(t))$. This leads to the following system of equations:

$$r_i(t) = \int_t^T e^{-(s-t)} [\lambda_i p_i(s) + (1 - \lambda_i) r_i(s)] ds, \tag{4.3a}$$

$$p_i(t) = m_i(r(t)). \tag{4.3b}$$

for all $i \in N$ and $t \in [0, T]$. It turns out that it has a unique solution.

Lemma 4.3.1. *System* (4.3) *has a unique solution* $(p^*, r^*) : [0, T] \to \mathbb{R}^n \times \mathbb{R}^n$.

Similar to Rubinstein (1982), a strategy profile can be constructed based on the solution to system (4.3).

Definition 24. (f^*, g^*) *is a strategy profile, such that for all* $i \in N$ *and for all* $t \in [0, T]$*:*

- *If i is proposing at time t, he offers $r_j^*(t)$ to all $j \neq i$, and claims $p_i^*(t)$ for himself.*
- *If i is responding at time t, he accepts a proposal v iff $v_i \geq r_i^*(t)$.*

(4)

An argument similar to Claim 3 of Ambrus and Lu (2015b) shows that (f^*, g^*) is the unique SPE of the game.

Proposition 4.3.2. *(f^*, g^*) is the unique SPE of the game Γ.*

4.4 Main Result

This section investigates the behavior of the payoffs (p^*, r^*) associated with the unique SPE (f^*, g^*) as the horizon T tends to infinity. Consider the game Γ from the previous section. Figure 4.3 shows the SPE payoffs of a player $i \in N$, as a function of the time $t \in [0, T]$.

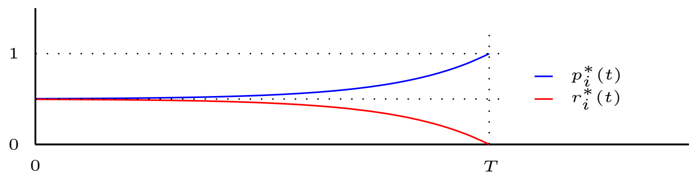


Figure 4.3. The SPE payoffs of a player $i \in N$.

The shape of these payoff curves is intuitive: the closer players get to the horizon T, the higher the probability the game will run out without another offer being made, and thus the higher the cost of rejecting a proposal. This means that, as the game approaches T, responders will have to accept lower offers, and proposers can make higher claims for themselves. This reasoning underlines the history-independent nature of the SPE: which proposals players make, or agree on, depends only on the time that remains until the game expires. It is thus useful to define functions x and y that specify players' SPE payoffs as a function of the remaining time to T. More specifically, $x(t) := r^*(T - t)$ and $y(t) := p^*(T - t)$. It is then sufficient to study the limit behavior of the functions x and y. To see this, suppose that after the start of the game, the first process realizes at some time $\bar{t} \in [0, T]$. Then the game concludes at $\bar{t}$ and the payoffs players realize are given by $x(T - \bar{t})$ and $y(T - \bar{t})$. Since the first arrival is exponentially distributed with a unit rate parameter, $\bar{t}$ is finite, meaning $(T - \bar{t})$ will tend to infinity with T.

The functions x and y are derived from system (4.3). In particular, by (4.3b)

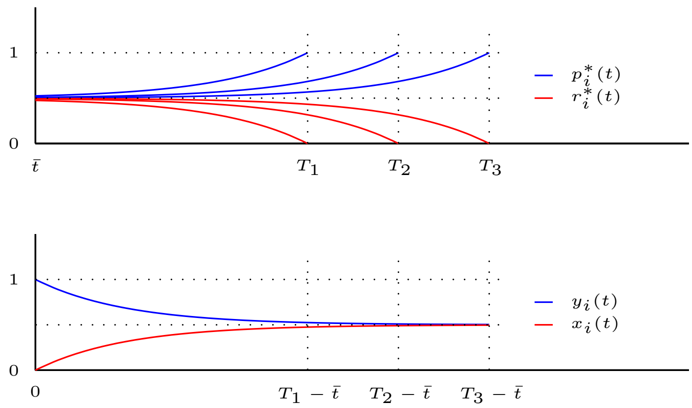


Figure 4.4. Time-$\bar{t}$ SPE payoffs with deadlines $T_3 > T_2 > T_1$.

$y(t) = m(x(t)) = m(r^*(T-t)) = p^*(T-t)$, and by (4.3a),

$$\begin{aligned} x_i(t) &= r_i^*(T-t) \\ &= \int_{T-t}^{T} e^{-(s-T+t)}[\lambda_i p_i^*(s) + (1-\lambda_i) r_i^*(s)]ds \\ &= \int_0^t e^{(\tau-t)}[\lambda_i p_i^*(T-\tau) + (1-\lambda_i) r_i^*(T-\tau)]d\tau \\ &= \int_0^t e^{(\tau-t)}[\lambda_i y_i(\tau) + (1-\lambda_i) x_i(\tau)]d\tau. \end{aligned}$$

for all $i \in N$. In the first place, this implies $x(0) = \mathbf{0}$. Furthermore, differentiating with respect to t yields

$$\frac{dx_i(t)}{dt} = \lambda_i(m_i(x(t)) - x_i(t)) \tag{4.5}$$

Thus, we obtain (4.2), the Initial Value Problem that defines the λ-weighted Raiffa solution, where $\lambda = (\lambda_1, \ldots, \lambda_n)$. It then follows from the definition that $x(T-\bar{t})$ converges to $R^\lambda(S)$ as T diverges. Since $y(t) = m(x(t))$ for all t, it follows from continuity of $m(\cdot)$ that $y(T-\bar{t})$ converges to $m(R^\lambda(S))$; since $R^\lambda(S) \in \partial S$, it follows that $y(T-\bar{t})$ converges to $R^\lambda(S)$ as well. Hence, also the payoff of the time-$\bar{t}$ proposer converges to the value implied by the λ-weighted Raiffa solution. We may summarize as follows.

Theorem 4.4.1. *The payoffs associated with the unique SPE of a game* $\Gamma = \{S, \lambda, T\}$ *converge to* $R^{\lambda}(S)$ *as the horizon* T *tends to infinity.*

Remark. There are two potential criticisms on Theorem 4.4.1. In the first place, it only provides *approximate* non-cooperative support for the Raiffa solution. More seriously, for every finite horizon, there is a strictly positive probability that bargaining ends before any player has the opportunity to make a proposal. In such a case players realize zero payoffs, without an action ever being played. However, using the approach of Trockel (2011), both criticisms may be tackled at once. In particular, consider an extension of the game Γ in which the deadline T is not exogenously specified, but rather, it is *chosen* by the first rejector of the first proposal. Then the game does not conclude before an offer is made, the first proposer proposes exactly the (weighted) Raiffa solution, and all opponents immediately accept. Hence, it yields an *exact* support result for R^{μ}.

Remark. Theorem 4.4.1 should not be confused with Theorem 2 of Ambrus and Lu (2015b). They consider a coalitional bargaining framework, where the underlying cooperative game is a convex TU game, while we take the underlying game to be a pure bargaining problem. Of course, for TU games where generating any surplus requires the grand coalition, Ambrus and Lu (2015b) also implement the continuous Raiffa solution, but the restriction to TU games means they only do so on the domain of bargaining problems for which the Pareto set is a linear transformation of the $(n-1)$-dimensional unit simplex. In such bargaining problems, the λ-weighted continuous Raiffa solution coincides with the λ-weighted discrete Raiffa solution, the λ-weighted Nash bargaining solution, the λ-weighted Kalai-Smorodinsky solution, and many others. The distinction arises in pure bargaining problems where utility is not transferable, indeed, the framework considered in this paper. This does not mean that this set-up is more general; as mentioned before, TU games and pure bargaining problems are simply different subsets of the class of NTU games.

4.5 An Alternative Proposer Protocol

Selten (1981) studied an elegant alternative proposer protocol in which the player who rejects the current proposal is called to make the next proposal. This protocol - also named the *rejector-proposes* protocol - has been studied primarily in the context of coalitional bargaining games, and has been shown to have potentially important implications for the resulting equilibria (see e.g. Chatterjee and K. (1993)). In this section it is demonstrated that this is not the case in the present game. That is, under the rejector-proposes protocol, the SPE is unchanged, SPE *payoffs* are unchanged, and these payoffs continue to converge to the Raiffa solution as the deadline of negotiations tends to infinity. This is in line with the findings of Britz et al. (2010), Britz and Predtetchinski (2012), who consider a bargaining game that provides non-cooperative support for the asymmetric Nash bargaining solution; whether the underlying protocol is action-independent or whether the designated next proposer is the last rejector, turns out to be immaterial to their support result.[8]

The Game As before, bargaining occurs in continuous time, in an interval that ranges from 0 to T with $T > 0$, the rate at which a player $i \in N$ can make proposals is governed by a Poisson process with player-specific arrival rate λ_i, and without loss of generality it is assumed that $\sum_i \lambda_i = 1$.

The main difference with respect to the game defined above is that players' processes no longer run concurrently. Instead, there is always a single designated next proposer who will make his offer at the first next arrival of his own Poisson process. It is assumed that player $\hat{i} \in N$ is the designated next proposer at time 0. A second departure from the previous game is that the proposer also votes on his own proposal, and moreover, that he is the first to do so. In particular, we assume that if player i puts an offer on the table, then the order of votes is given by $[i, i+1, \ldots, n, 1, \ldots, i-1]$ if $i > 1$, and by $[1, \ldots, n]$ otherwise.[9] Unanimous agreement on a proposal continues to end the game with the implementation of that proposal. However, if unanimous agreement is not reached, then the game continues, but now with the first rejector in the above-defined order in the role of designated next proposer. As before, if no unanimous agreement is reached before

[8] Britz and Predtetchinski (2012) in fact study a more general action-dependent proposer protocol that includes the rejector-proposes protocol as a special case.

[9] While the exact order of the players deciding after the proposer continues to be irrelevant, it is essential that the proposer decides on his own offer first.

or at time T, then players realize the disagreement outcome $\mathbf{0}$. Such a game is denoted by $\Gamma^{\text{RP}} = \{S, \lambda, N\}$, where S, λ and T are as defined above.

Strategies In a game Γ^{RP}, strategies are somewhat simpler to define. In particular, rather than specifying the identities of all previous proposers, histories only need to specify who is the first proposer. The identities of all subsequent proposers can then be inferred from the play of the game. Thus, a history in H_i^p specifies the first designated next proposer $\hat{i} \in N$, and further specifies for each $t \in (0, T]$ the times $0 \leq t_1 \leq \ldots \leq t_k < t$ of all previous proposals (if any), the corresponding proposals, and the corresponding sets of rejectors. A history in H_i^r additionally specifies the time-t proposal, and the set of rejectors prior to i. A strategy for player i is again a pair of functions (f_i, g_i) with $f_i : H_i^p \to S$ and $g_i : H_i^r \to \{Y, N\}$; a strategy profile is again denoted by (f, g).

Subgame Perfect Equilibrium A heuristic reasoning is helpful in the construction of an SPE. Consider a game Γ^{RP}, and assume that players play a strategy profile (f, g), such that proposers make all opponents indifferent between accepting and rejecting, and furthermore that such proposals are immediately accepted. Suppose a player i's associated time-t (expected) payoff, given that player $j \in N$ is called to be the next proposer, is denoted $q_i^j(t)$. Since the first rejector is called to be the next proposer, and since the proposal prescribed by f is assumed to be accepted by all, deviating leads to the payoff $r_i(t) := q_i^i(t)$. Hence, if player i is the proposer at time t, he offers $r_j(t)$ to all opponents $j \neq i$. Denoting the payoff i realizes himself as $p_i(t)$, this yields

$$r_i(t) = \int_t^T \lambda_i e^{-\lambda_i(s-t)} p_i(s) ds \tag{4.6a}$$

$$p_i(t) = m_i(r(t)) \tag{4.6b}$$

for all $i = 1, \ldots, n$ and $t \in [0, T]$. Observe that $r_i(T) = 0$. Furthermore, differentiating equation (4.6a) yields $r_i'(t) = -\lambda_i p_i(t) + \lambda_i r_i(t)$. Thus, the solutions to systems (4.3) and (4.6) coincide.

Proposition 4.5.1. *System* (4.6) *has a unique solution* $(\hat{p}, \hat{r}) : [0, T] \to \mathbb{R}^n \times \mathbb{R}^n$. *Moreover,* $(\hat{p}, \hat{r}) = (p^*, r^*)$.

This allows for defining a strategy profile $(\hat{f}, \hat{g})$ that is analogous to (f^*, g^*). In particular, given a time-t history in H_i^p, a proposer i's strategy $\hat{f}_i$ is to propose $(p_i^*(t), r_{-i}^*(t))$; given a time-t history in H_i^r that includes the time-t proposal $v \in S$, a responder i's strategy $\hat{g}_i$ is to accept v if and only if $v_i \geq r_i^*(t)$. An argument analogous to Proposition 4.3.2 then demonstrates that the change of protocol has no influence on the outcome of the game.

Proposition 4.5.2. *$(\hat{f}, \hat{g})$ is the unique SPE in the game Γ^{RP}.*

4.6 Concluding Remarks

This paper has provided a non-cooperative foundation for the continuous Raiffa solution in multilateral bargaining problems. Moreover we showed that this foundation does not rely on the convexity of the bargaining set. While the game introduced to this end is rather natural, it does include the somewhat unrealistic assumption that players do not discount their payoffs over time. A natural extension would thus be to allow for the discounting of utilities. In this case, a connection seems to arise with the Nash bargaining solution. It may in the first place be conjectured that the game enriched with a discount factor continues to have a unique SPE in stationary strategies, and that the associated SPE payoffs continue to converge to a stationary point as the deadline T tends to infinity. Given that the feasible set satisfies a certain smoothness condition, this stationary point will in fact converge to the Nash bargaining solution as the discount factor tends to one (see e.g. Kultti and Vartiainen (2010)). Exploring the modified version of the game with a discount factor in more detail is left for future work.

.1 Proofs

.1.1 Proof of Proposition 4.2.1

Consider $S \in \mathcal{B}$, and let $\|\cdot\|$ denote the taxicab metric – i.e., for $x \in \mathbb{R}^n$, $\|x\| := \sum_{i=1}^n |x_i|$. Then we have the following useful lemma.

Lemma .1.1. *The function $m(\cdot)$ is Lipschitz continuous on $D := \{z \in S \mid z' \leq z \leq z''$ where $z', z'' \in S \cap \mathbb{R}^n_+$ with $z' < z''\}$, with Lipschitz constant $L = nK$.*

Proof. Consider $v, w \in D$, and define the sequence $\{z^k\}_{k=0}^n$ where $z^0 := v$, and for all $k = 1, \ldots, n$, $z^k := (w_1, \ldots, w_k, v_{k+1}, \ldots, v_n)$. Then each z^k is an element of D, and z^k and z^{k-1} only differ in the k-th coordinate. For $k \in N$ and $i \in N \setminus \{k\}$, we have that the points $(m_i(z^k), z^k_{-i})$ and $(m_i(z^{k-1}), z^{k-1}_{-i})$ are in $\partial S \cap \mathbb{R}^n_+$. Since $S \in \mathcal{B}$, it then follows from condition (A1) that

$$|m_i(z^k) - m_i(z^{k-1})| \leq K|z^k_k - z^{k-1}_k| = K|w_k - v_k|.$$

In addition, $|m_k(z^k) - m_k(z^{k-1})| = 0 \leq K|w_k - v_k|$. Then for all $i \in N$ we have

$$\begin{aligned} |m_i(w) - m_i(v)| &= \left|\sum_{k=1}^n m_i(z^k) - m_i(z^{k-1})\right| \leq \sum_{k=1}^n |m_i(z^k) - m_i(z^{k-1})| \\ &\leq \sum_{k=1}^n K|w_k - v_k| = K\|w - v\|. \end{aligned}$$

Therefore, $\|m(w) - m(v)\| = \sum_{i=1}^n |m_i(w) - m_i(v)| \leq nK\|w - v\|$. □

Fix some $\mu \in \mathbb{R}^n_{++}$ and define $f(x) := \mu(m(x) - x)$. By Lemma .1.1, the function f satisfies a uniform Lipschitz condition on $S \cap \mathbb{R}^n_+$. Then by the Picard-Lindelöf theorem it follows that problem (4.2) has a unique solution $x(t)$. Consider the maximal interval of existence $[0, \omega)$ of this solution.

(i) For all $z \in \text{int}(S) \cap \mathbb{R}^n_+$, $f(z) > \mathbf{0}$. Hence, $x(t)$ is a strictly increasing function. Then by the extension theorem (e.g. Theorem 8.33 of Kelley and Peterson (2010)), it follows that $x(t)$ converges to a point $\hat{x} \in \partial S \cap \mathbb{R}^n_+$ as $t \to \omega$.

(ii) Assume that ω is finite. Then the function $v(t) := x(\omega - t)$ is the unique solution to the problem $[v'(t) = -f(v)$ and $v(0) = \hat{x}]$. Since $v'(0) = \mathbf{0}$, it follows that $v(t) = \hat{x}$ for all $t \in [0, \omega]$. Since this implies $\mathbf{0} = x(0) = v(\omega) = \hat{x}$ – a contradiction – it follows that $\omega = \infty$. □

.1.2 Proof of Lemma 4.3.1

The first equation of the obtained system (4.3) can be equivalently written as a differential equation. In particular, for $i \in N$ we have

$$\begin{aligned}\frac{dr_i(t)}{dt} &= 0 - \frac{dt}{dt}e^{-(t-t)}[\lambda_i p_i(t) + (1-\lambda_i)r_i(t)] \\ &\qquad + \int_t^T e^{-(s-t)}[\lambda_i p_i(s) + (1-\lambda_i)r_i(s)]ds \\ &= -[\lambda_i p_i(t) + (1-\lambda_i)r_i(t)] + r_i(t) \\ &= -\lambda_i m_i(r(t), S) + \lambda_i r_i(t)\end{aligned}$$

Furthermore, $r(T) = \bar{0}$. Then the proof follows along the lines of Proposition 4.2.1. □

.1.3 Proof of Proposition 4.3.2

Consider the game Γ, and for $t \in [0, T]$, let $\overline{r}_i(t)$ and $\underline{r}_i(t)$ respectively be player i's associated supremum and infimum reservation values over all SPE's, and all time-t histories in H_i^r. Assume that

$$\underline{r}(t) = \overline{r}(t) = r^*(t) \quad \text{and} \quad r^*(t) \in \mathrm{int}(S) \quad \text{for all } t \in [\hat{T}, T], \tag{7}$$

where $\hat{T}$ is some time in $(0, T]$, r^* represents players' reservation values under strategy profile (f^*, g^*), and $\mathrm{int}(S)$ again denotes the interior of S. The aim of the proof is to show that (7) also holds on a non-trivial time interval that precedes $\hat{T}$. The following lemma helps define that interval.

Lemma .1.2. *There exists a $c > 0$ such that for all $x \in int(S) \cap \mathbb{R}^n_+$, we have that $x + c(m(x) - x) \in S$.*

Proof. Let $x \in \mathrm{int}(S) \cap \mathbb{R}^n_+$, let $y := x + \alpha^*(m(x) - x)$ where $\alpha^* := \max\{\alpha \mid x + \alpha(m(x) - x) \in S\}$, and let $F := \{z \in \mathbb{R}^n \mid x \le z \le y\}$. By Lemma .1.1 it follows that for all $v, w \in F$, we have

$$\|(m(w) - w) - (m(v) - v)\| \le (1 + nK)\|w - v\|. \tag{8}$$

Note that $m(y) = y$, so $\|(m(x) - x) - (m(y) - y)\| = \|m(x) - x\|$. Furthermore, $\|x - y\| = \|x - x - \alpha^*(m(x) - x)\| = \alpha^*\|m(x) - x\|$. Then by (8), it follows that

$$\|m(x) - x\| \leq \alpha^*(1 + nK)\|m(x) - x\|.$$

Since $x \in \mathrm{int}(S) \cap \mathbb{R}^n_+$, this implies $\alpha^* \geq \frac{1}{1+nK}$. It follows that for all $x \in \mathrm{int}(S)$, $x + \frac{1}{1+nK}(m(x) - x) \in S$. $\square$

Let $\tau := \ln \frac{2}{2+c}$, where c is as in Lemma .1.2. Then the probability of another arrival in the interval $[\hat{T} - \tau, \hat{T}]$ is given by $c/2$

Lemma .1.3. *For all $t \in [\hat{T} - \tau, \hat{T}]$ we have $\overline{r}(t) \in int(S)$.*

Proof. Let v be an SPE proposal accepted at a time $t \in [\hat{T} - \tau, \hat{T}]$. Any player $i \in N$ can secure the payoff $r_i^*(\hat{T})$ by

(i) rejecting the proposal v at time t and all subsequent proposals at times $t' \in (t, \hat{T}]$, and

(ii) claiming $r_i^*(\hat{T})$ at any time $t' \in [t, \hat{T}]$ where he himself is the proposer.

Then SPE implies that $v \geq r^*(\hat{T})$. Assume next that there is a player $i \in N$ with $v_i > m_i(r^*(\hat{T}))$. Since then there is a player $j \neq i$ for whom $v_j < r_j^*(\hat{T})$, contradicting the above, it follows that $v \leq m(r^*(\hat{T}))$. It follows that expected payoffs in SPE are bounded between $r^*(\hat{T})$ and $m(r^*(\hat{T}))$.

Consider a player i and a time $t \in [\hat{T} - \tau, \hat{T}]$. If no more arrivals occur within the interval $[t, \hat{T}]$, player i realizes the payoff $r_i^*(\hat{T})$; if on the other hand a process *does* realize within this interval, then i's payoff is bounded above by $m_i(r^*(\hat{T}))$. The former occurs with a probability $e^{-(\hat{T}-t)}$, and the latter with probability $(1 - e^{-(\hat{T}-t)})$. Hence,

$$\begin{aligned}
\overline{r}_i(t) &\leq e^{-(\hat{T}-t)} r_i^*(\hat{T}) + (1 - e^{-(\hat{T}-t)}) m_i(r^*(\hat{T})) \\
&= r_i^*(\hat{T}) + (1 - e^{-(\hat{T}-t)})(m_i(r^*(\hat{T})) - r_i^*(\hat{T})) \\
&\leq r_i^*(\hat{T}) + \frac{c}{2}(m_i(r^*(\hat{T})) - r_i^*(\hat{T})) \\
&< r_i^*(\hat{T}) + c(m_i(r^*(\hat{T})) - r_i^*(\hat{T}))
\end{aligned}$$

The lemma then follows by Lemma .1.2 and comprehensiveness of S. $\square$

Lemma .1.4. *For all $t \in [\hat{T} - \tau, \hat{T}]$ and $i \in N$ we have*

$$\underline{r}_i(t) = \int_t^{\hat{T}} e^{-(s-t)}[\lambda_i m_i(\overline{r}(s)) + (1-\lambda_i)\underline{r}_i(s)]ds + r_i^*(\hat{T}) \quad \textit{and} \tag{9a}$$

$$\overline{r}_i(t) = \int_t^{\hat{T}} e^{-(s-t)}[\lambda_i m_i(\underline{r}(s)) + (1-\lambda_i)\overline{r}_i(s)]ds + r_i^*(\hat{T}). \tag{9b}$$

Proof. Consider a player i, and define the following strategy profile. For all $t \in [0, T]$:

- Player i offers $\overline{r}_j(t)$ to all $j \neq i$, and claims $m_i(\overline{r}(t))$ himself. Players $j \neq i$ accept an offer v iff $v_j \geq \overline{r}_j(t)$. In case of disagreement, the game moves to an SPE in which the first rejector j receives $\underline{r}_j(t)$.

- Player $k \neq i$ offers $\underline{r}_i(t)$ to player i, $\overline{r}_j(t)$ to all $j \neq i, k$, and claims $m_k(\underline{r}_i(t), \overline{r}_{-i}(t))$ for himself. Player i accepts a proposal v iff $v_i \geq \underline{r}_i(t)$, player $j \neq i, k$ accepts v iff $v_j \geq \overline{r}_j(t)$. In case k's proposal is rejected, then the game moves on to an SPE in which first rejector j realizes the payoff $\underline{r}_j(t)$.

It follows from Lemma .1.3 that for $t \in [\hat{T} - \tau, \hat{T}]$, the above proposals are feasible. It is further immediate that responders in this strategy profile cannot profitably deviate from their strategies. To see that the same is true for proposers, consider first the case where i is proposing. By Lemma .1.3 and the definition of $m(\cdot)$, it follows that $\overline{r}(t) < m(\overline{r}(t))$, and thus $\overline{r}_i(t) < m_i(\overline{r}(t))$; it follows that player i has no profitable deviation. Consider a proposer $k \neq i$, and observe that $\overline{r}_k(t) < m_k(\overline{r}(t))$ by similar reasoning as before. Moreover, $m_k(\overline{r}(t)) \leq m_k(\overline{r}_i(t), \underline{r}_i(t))$. If he deviates from the above strategy, then k's expected payoff is dominated by $\overline{r}_k(t)$, and thus by $m_k(\overline{r}_i(t), \underline{r}_i(t))$. Hence, player k cannot profitably deviate either.

It follows that for all $t \in [\hat{T} - \tau, \hat{T}]$, we have

$$\begin{aligned}
\underline{r}_i(t) &= \int_t^T e^{-(s-t)}[\lambda_i m_i(\overline{r}(s)) + (1-\lambda_i)\underline{r}_i(s)]ds \\
&= \int_t^{\hat{T}} e^{-(s-t)}[\lambda_i m_i(\overline{r}(s)) + (1-\lambda_i)\underline{r}_i(s)]ds \\
&\qquad + \int_{\hat{T}}^T e^{-(s-t)}[\lambda_i m_i(r^*(s)) + (1-\lambda_i)r_i^*(s)]ds \\
&= \int_t^{\hat{T}} e^{-(s-t)}[\lambda_i m_i(\overline{r}(s)) + (1-\lambda_i)\underline{r}_i(s)]ds + r_i^*(\hat{T}).
\end{aligned}$$

All other cases are similar. □

From Lemma 4.3.1 it follows that $(\overline{r}(t), \underline{r}(t)) = (r^*(t), r^*(t))$ is the *unique* solution to system (9). Hence, condition (7) holds on the interval $[\hat{T} - \tau, T]$.

Observe that $\underline{r}(T) = \overline{r}(T) = \bar{0} = r^*(T)$ and $\mathbf{0} \in \text{int}(S)$. Thus, since $\tau > 0$ and T is finite, iteratively applying the above argument leads in a finite number of steps to the conclusion that $\underline{r}(t) = \overline{r}(t) = r^*(t)$ for all $t \in [0, T]$. It then follows that the strategy profile (f^*, g^*) is the unique SPE of the game Γ. □

.1.4 Proof of Proposition 4.5.2

The argument is similar to Proposition 4.3.2. Attention is focused on the steps that differ.

- Let $\overline{r}_i(t)$ and $\underline{r}_i(t)$ be player i's supremum respectively infimum expected payoff over all SPE's, and all time-t histories in H_i^r that are such that, player i is the designated next proposer. Again, assume that for all $t \in [\hat{T}, T]$, we have $\underline{r}(t) = \overline{r}(t) = r^*(t)$, and that $r^*(t)$ is contained in int(S).

- Consider $c > 0$ as defined in Lemma .1.2, but now define $\tau := \ln\left(\frac{2}{2+c}\right)^{\frac{1}{\tilde{\lambda}}}$ where $\tilde{\lambda} := \max_{i \in N} \lambda_i$. Then whichever player is called to be the next proposer, the probability of another arrival in the interval $[\hat{T} - \tau, \hat{T}]$ is at most $c/2$.

- Consider an SPE in which a proposal v is accepted at some time $t \in [\hat{T} - \tau, \hat{T}]$. Suppose that $v_i > m_i(r^*(\hat{T}))$ for some player $i \in N$. Then there is a player

$j \in N \setminus i$ for whom $v_j < r_j^*(\hat{T})$. As before, such a player can profitably deviate from his strategy. In particular, he can realize $r_j^*(\hat{T})$ by rejecting v at time t, becoming the designated next proposer, and rejecting each offer he subsequently gets to make in the interval $(t, \hat{T}]$. This again implies that any *expected* SPE payoff in the interval $[\hat{T} - \tau, \hat{T}]$ is below $m(r^*(\hat{T}))$.

Suppose that player i is called to be the next proposer at a time $t \in [\hat{T} - \tau, \hat{T}]$. The probability of his process realizing within the interval $[t, \hat{T}]$ is $(1 - e^{-\lambda_i(\hat{T}-t)})$, and the probability that no arrival occurs is $e^{-\lambda_i(\hat{T}-t)}$. In the former case he realizes a payoff of at most $m_i(r^*(\hat{T}))$, in the latter case he realizes a payoff in excess of $r_i^*(\hat{T})$. By the same reasoning as in Proposition 4.3.2 we then obtain $\overline{r}_i(t) < r_i^*(\hat{T}) + c(m_i(r^*(\hat{T})) - r_i^*(\hat{T}))$, which by Lemma .1.2 and comprehensiveness implies $\overline{r}(t) \in \text{int}(S)$.

- Analogous to Proposition 4.3.2 we can define $2n$ strategy profiles, feasible and optimal in the interval $[\hat{T} - \tau, \hat{T}]$, such that

$$\underline{r}_i(t) = \int_t^{\hat{T}} \lambda_i e^{-\lambda_i(s-t)} m_i(\overline{r}(s))ds + r_i^*(\hat{T}) \quad \text{and} \tag{10a}$$

$$\overline{r}_i(t) = \int_t^{\hat{T}} \lambda_i e^{-\lambda_i(s-t)} m_i(\underline{r}(s))ds + r_i^*(\hat{T}). \tag{10b}$$

for all $i \in N$. The unique solution of this system on the interval $[\hat{T} - \tau, \hat{T}]$ is again $(\underline{r}(t), \overline{r}(t)) = (r^*(t), r^*(t))$. As before, a finite number of iterations of this argument leads to the conclusion that $r^*(t)$ is the unique solution on the entire interval $[0, T]$, and thus that the strategy profile $(\hat{f}, \hat{g})$ is the unique SPE of the game. $\square$

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